Lecture 3: Stochastic processes and nowhere differentiable functions

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Contents

Introduction

- Options derive their values V from prices of the underlying stock S (hence the term 'derivatives').
- The pricing problem is a problem, only because the future stock price $S(T)$ is unknown.
- Can we predict the future stock price $S(T)$ (e.g. by using historical stock prices $S(t)$ up to the current time moment $t < T$)?
- We begin by discussing a naive attempt to predict stock prices using classical *differential calculus* (i.e. like in celestial mechanics), and then explain why these methods fail in general.
- This will provide the main motivation for the *stochastic calculus*, on which the Black-Scholes pricing theory is based.
- We shall define the concept of a stochastic process (aka random function) and define their main characteristics, such as the power spectrum and correlation time.

1 Signals and Processes in Time

Signals and Processes in Time

Definition 1 (Process (Signal)). $x(t)$ is a sequence of values $x(0), x(t_1), x(t_2), \dots x(T)$, indexed by an increasing sequence $0 < t_1 < t_2 < \dots < T$ of time moments, that conveys information about the states of a system or a phenomenon at these time moments $t \in [0, T]$. Thus, s process is a *function* of time $x(t)$, the domain of which is time $t \in [0, T]$ and the range $x \in X$ is some variable of interest. Typical notation:

$$\{x(t)\}_{t\in[0,T]}$$

Example 2.

- Position $x(t)$ of a car at different time moments.
- Records of temperature $\tau(t)$ in London on different days.
- Prices of stock $S(t)$ at times $t_1, t_2, \dots$.
- Sequence of outcomes of tossing a coin (e.g. "tail", "head", "head").

Discrete or Continuous

Definition 3 (Discrete or continuous-time processes). A process is said to be

- *discrete-time* process, if its values $x(t)$ are defined only for a discrete set of time moments $t_1, t_2, \dots$ (i.e. countable domain).
- *continuous-time* process, if $x(t)$ are defined for all time moments on the interval $[0, T] \subseteq \mathbb{R}$ (uncountable domain).

Question 1. *In the examples above, which of the processes are discrete and which are continuous time?*

Definition 4 (Discrete or continuous-valued processes). A process is said to be

- *discrete-valued*, if the values $x(t)$ are elements of a countable set X (countable range).
- *continuous-valued*, if the values $x(t)$ are elements of a continuous (uncountable) set X (uncountable range).

Question 2. *In the examples above, which of the processes are discrete and which are continuous-valued?*

Prediction of differentiable processes

Predicting position of a car

- If $x(0)$ is the initial position of a car moving with constant speed v, then position of the car $x(t)$ for any $t \in [0, T]$ is

$$x(t) = x(0) + vt$$

- If the car is accelerating from $v(0)$ with constant acceleration a, then

$$x(t) = x(0) + v(0)t + \frac{1}{2}at^2$$

- Note that $v = \dot{x} = dx/dt$ and $a = \ddot{x} = dv/dt = d^2x/dt^2$:

$$x(t) = x(0) + \dot{x}(0)t + \frac{1}{2}\ddot{x}t^2$$

- Could the same logic be applied to predict more general processes (e.g. stock prices $S(t)$)?

Differentiation

Definition 5. A function $x(t)$, that is continuous at t_0, has a derivative $x'(t_0)$ at t_0 if the following limit exists:

$$x'(t_0) = \lim_{\Delta t \to 0} \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t}$$

- Other notations

$$\frac{dx(t)}{dt}, \quad \dot{x}$$

- $x' = dx/dt$ represents the *slope* of a tangent line to $x(t)$ passing through $(t_0, x(t_0))$.
- The tangent line gives the best linear approximation:

$$x(t) \approx x(t_0) + x'(t_0)(t - t_0)$$

Taylor Expansion

Taylor series expansion

- If $x(t)$ is n times differentiable at t_0 (i.e. derivatives x', x'',..., $x^{(n)}$ exist at t_0), then $x(t)$ can be approximated as follows:

$$x(t) \approx x(t_0) + x'(t_0)(t - t_0) + \frac{x''(t_0)}{2}(t - t_0)^2 + \cdots + \frac{x^{(n)}(t_0)}{n!}(t - t_0)^n$$

- If $x(t)$ is infinitely many times differentiable at t_0, then $x(t)$ can be found precisely by Taylor expansion at t_0:

$$x(t) = \sum_{n=0}^{\infty} \frac{x^{(n)}(t_0)}{n!}(t - t_0)^n$$

- How likely does the stock price process $S(t)$ have any derivatives?

Nowhere differentiable functions

- The existence of derivative $x'(t)$ implies continuity of $x(t)$ (Why? Recall the definitions of $x'(t)$ and of continuity of a function).
- Is the opposite true (i.e. does continuity imply differentiability)?
- Simple answer is *NO* (e.g. $x(t) = |t|$ has no $x'(0)$).
- In 1872, Karl Weierstrass gave an example of a function that is continuous everywhere, but differentiable *nowhere*:

$$x(t) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi t)$$

- Rather than being exotic, it was proved by Banach and Mazurkiewics that almost all continuous functions are nowhere differentiable.
- It turns out that a trajectory $w(t)$ of Brownian motion is another such example, because $\Delta w(t) = w(t + \Delta t) - w(t)$ is proportional to $\sqrt{\Delta t}$, so that:

$$\frac{w(t + \Delta t) - w(t)}{\Delta t} \sim \frac{\sqrt{\Delta t}}{\Delta t} = \frac{1}{\sqrt{\Delta t}}$$

2 Stochastic Processes

Stochastic Process

Definition 6 (Stochastic Process)**.** Process $\{x(t)\}_{t\in[0,T]}$ is *stochastic* if for each time moment $t \in [0, T]$ its value is a random variable $x(t, \omega)$, defined by a probability space $(\Omega, \mathcal{A}, P)$ (or by the probability density $p(t, x) = dP(x(t, \omega))/dx$). Thus, a stochastic process is a *random function* $x(t, \omega)$.

Example 7 (Probability densitiy).

- Uniform density:

$$p(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

- Gaussian density:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$

  where $\bar{x} = \mathbb{E}_p\{x\} = \int x p(x)\, dx$ is the *mean* and $\sigma^2 = \mathbb{E}_p\{(x - \bar{x})^2\} = \mathbb{E}_p\{x^2\} - \bar{x}^2$ is the *variance*.

Description of a Continuous-time Stochastic Process

- For each finite partition $0 < t_1 < \cdots < t_n < T$ of $[0,T]$, the values $x(t_1),\ldots,x(t_n)$ constitute a finite collection of random variables defined by joint probability density $p(x(t_1),\ldots,x(t_n))$ such that:

$$p(x(t_1),\ldots,x(t_n)) = \int p(x(t_1),\ldots,x(t_n),x(t_{n+1}))\,dx(t_{n+1})$$

- Thus, the $n+1$-dimensional partition $0 < t_1 < \cdots < t_{n+1} < T$ and density includes all the information about n-dimensional partition $0 < t_1 < \cdots < t_n < T$ and density.

- The complete information about continuous-time stochastic process $\{x_t\}_{t\in(0,T)}$ is given by the *probability functional*:

$$P[x(t)] = \lim_{\max \Delta t\to 0} p(x(t_1),\ldots,x(t_n))$$

- Stochastic process is *stationary*, if its characteristics at t and $t+\tau$ are the same (translation invariant):

$$p(x(t_1),\ldots,x(t_n)) = p(x(t_1+\tau),\ldots,x(t_n+\tau))$$

Characteristic Functional

- Alternatively, a stochastic process can be characterised by the *characteristic functional*:

$$\begin{aligned}
\Theta[u(t)] &= \mathbb{E}_P\left\{\exp\left[i\int u(t)x(t)\,dt\right]\right\}\\
&= 1+\sum_{s=1}^{\infty}\frac{i^s}{s!}\int\cdots\int m_s(t_1,\ldots,t_s)\,u(t_1)\ldots u(t_s)\,dt_1\ldots dt_s\\
&= \exp\left\{\sum_{s=1}^{\infty}\frac{i^s}{s!}\int\cdots\int k_s(t_1,\ldots,t_s)\,u(t_1)\ldots u(t_s)\,dt_1\ldots dt_s\right\}
\end{aligned}$$

- The *moment* functions $m_s(t_1,\ldots,t_s) = \mathbb{E}\{x(t_1)\cdots x(t_s)\}$ and the *correlation* functions $k_s(t_1,\ldots,t_s)$ are the derivatives of $\Theta[u]$:

$$m_s(t_1,\ldots,t_s) = \frac{1}{i^s}\frac{\partial^s\Theta(u_1,\ldots,u_s)}{\partial u_1\cdots\partial u_s}\bigg|_{u=0}\,,\quad k_s(t_1,\ldots,t_s) = \frac{1}{i^s}\frac{\partial^s\ln\Theta(u_1,\ldots,u_s)}{\partial u_1\cdots\partial u_s}\bigg|_{u=0}$$

Gaussian stochastic process

- If for arbitrary partition $\{t_1,\ldots,t_n\}\subset(0,T)$, the density of $\{x_1,\ldots,x_n\}$ is Gaussian:

$$p(x_1,\ldots,x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\det\|k_{ij}\|}}e^{-\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x_i-\bar{x}_i)(x_j-\bar{x}_j)}$$

where $\bar{x}_i = \mathbb{E}\{x_i\}$ are the *mean* values and

$$k_{ij} = \mathbb{E}\{(x_i - \bar{x}_i)(x_j - \bar{x}_j)\} = \mathbb{E}\{x_i x_j\} - \bar{x}_i \bar{x}_j$$

are the *covariances*. They completely define a Gaussian process.

- The matrix $\|a_{ij}\|$ is the inverse $\|k_{ij}\|^{-1}$ of the covariance matrix.

Example 8.

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - R_{12}^2}} e^{-\frac{1}{2(1-R_{12}^2)}\left[\frac{(x_1-\bar{x}_1)^2}{\sigma_1^2} + \frac{(x_2-\bar{x}_2)^2}{\sigma_2^2} + 2R_{12}\frac{(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{\sigma_1\sigma_2}\right]}$$

where $R_{12} = \frac{k_{12}}{\sigma_1\sigma_2}$ is the correlation coefficient.

3 Characteristics of Stochastic Processes

Correlation function

- Correlation (or auto-correlation) function:

$$k(t, t+\tau) = \mathbb{E}\{x(t)x(t+\tau)\} - \mathbb{E}\{x(t)\}\mathbb{E}\{x(t+\tau)\}$$

- For a stationary process $k(t, t+\tau) = k(0, \tau) =: k(\tau)$:

$$k(\tau) := k(0, \tau) = \mathbb{E}\{x(t)x(t+\tau)\} - \mathbb{E}^2\{x\}$$

- Some properties:

$$k(0) = \sigma^2\,, \qquad k(\tau) = k(-\tau)$$

Correlation time

- Correlation time is defined as the following the integral of the correlation function:

$$\tau_{\text{cor}} = \frac{1}{\sigma^2}\int_0^\infty |k(\tau)|\, d\tau$$

- Note that $\sigma^2 = k(0)$.
- Because for stationary processes $\sigma^2(t) = \sigma^2(t+\tau)$, we can re-write this using correlation coefficient $R(\tau) = k(\tau)/\sigma^2$:

$$\tau_{\text{cor}} = \int_0^\infty |R(\tau)|\, d\tau$$

Spectral density (power spectrum)

- The *spectral density* $S[\lambda]$ (or the *power spectrum*) of stochastic process $\{x(t)\}_{t\in(0,T)}$ is the Fourier transform $S[\lambda] = \mathcal{F}[k(\tau)]$ of its correlation function:
$$S[\lambda] = \int_{-\infty}^{\infty} k(\tau)\, e^{-i\lambda\tau}\, d\tau$$
- The inverse transform recovers the correlation function:
$$k(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S[\lambda]e^{i\lambda\tau}\, d\lambda$$
- Another characterisation of the variance:
$$\sigma^2 = k(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S[\lambda]\, d\lambda$$

Reading

- Chapter 6, Sec. 6.1 (Elliott & Kopp, 2004).
- Chapter 9 (Roman, 2012).
- Chapter 1, Sec. 3 (Stratonovich, 2014).

References

Elliott, R. J., & Kopp, P. E. (2004). *Mathematics of financial markets* (2nd ed.). Springer.

Roman, S. (2012). *Introduction to the mathematics of finance: Arbitrage and option pricing*. Springer.

Stratonovich, R. L. (2014). *Topics in the theory of random noise* (Vol. 1). Martino Fine Books.