Lecture 6: Introduction to Set Theory

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BIS4410

Contents

The boy gave the girl the flower.
only

Only the boy gave the girl the flower.
The *only* boy gave the girl the flower.
The boy *only* gave the girl the flower.
The boy gave *only* the girl the flower.
The boy gave the *only* girl the flower.
The boy gave the girl *only* the flower.
The boy gave the girl the *only* flower.
The boy gave the girl the flower *only*.

1 Sets and Operations on Sets

Sets

Definition 1 (Set)**.**

- A collection of objects, called *elements* of the set.
- Sets are denoted by upper case letters (e.g. $A, B, C, \dots$), while their elements are denoted by lower case letters (e.g. $a, b, c, \dots$).
- $a \in A$ means 'a belongs to set A', and $b \notin A$ means b does not.

Example 2. Let $A = \{\text{apple}, \text{orange}, \text{lemon}\}$. Then 'orange' $\in A$ and 'cucumber' $\notin A$.

Example 3. Let $X = \{x_1, \dots, x_{10}\}$. Then $x_2 \in X$ and $x_{100} \notin X$.

How to Write Sets

1. Enumerating elements of the set:

$$A = \{a, b, c\}\,, \qquad B = \{1, 3, 5, 7, 9\}$$

2. Set comprehension notation:

$$A = \{a : P(a)\}$$

where : means '*such that*', and $P(x)$ is a rule that describes the common property.

Definition 4 (Set)**.** is a collection of objects (elements) of some kind with a common property P, so that, given an object a and a set A, it is possible to decide if the object belongs to the set or not (i.e. $P(a)$ is true or false).

Example 5. Set B can be written $B = \{b : b \text{ is an odd number and } 0 < b < 10\}$

Cardinality of a Set

Definition 6 (Cardinality)**.** of set A is the number of its elements, and denoted $|A|$.

Example 7. Let $A = \{\text{apple}, \text{orange}, \text{lemon}\}$. Then $|A| = 3$.

- A set $\{\}$ with no elements is called *empty set*, and denoted $\varnothing$.
- There are *infinite* sets, such as the set of all natural numbers:

$$\mathbb{N} := \{1, 2, 3, 4, \dots\}$$

- There are infinite sets with different cardinalities (i.e. different 'kinds' of infinity). For example, the set $\mathbb{R}$ of all real numbers is strictly larger than $\mathbb{N}$.

Question 1. *What is the cardinality of $\varnothing$? Of $\{\varnothing\}$? Is $\varnothing = \{\varnothing\}$?*

Subsets

Definition 8 (Subset)**.** of set B is set A such that every element of A is also an element of B. Denoted:

$$A \subseteq B \quad \text{or} \quad A \subset B$$

$\subset$ subset (proper)

$$\{a, b, c\} \subset \{a, b, c, d\}$$

$\subseteq$ subset or equal

$$\{a, b, c\} \subseteq \{a, b, c\}$$

$\equiv$ equivalence of sets

$$A \equiv B \quad \text{iff} \quad A \subseteq B \text{ and } A \supseteq B$$

Remark 1.
- *Empty set $\varnothing$ is a subset of any other set $\varnothing \subseteq A$.*
- *Any set is a subset of itself $A \subseteq A$.*

Power Set

Definition 9 (Power Set)**.** of set A is the set of all subsets of A. Denoted $\mathcal{P}(A)$ or 2^A.

Example 10. Power set of $\{0, 1\}$ is $\{\varnothing, \{0\}, \{1\}, \{0, 1\}\}$

Question 2. *What is the power set of $A = \{apple, orange, lemon\}$?*

Remark 2. *Cardinality of power set 2^A is $2^{|A|}$.*

Set Unions and Intersections

Definition 11 (Union $A \cup B$)**.** is the set of all elements that belong to A or B:

$$\{a, b, c\} \cup \{b, c, d\} = \{a, b, c, d\}$$

Definition 12 (Intersection $A \cap B$)**.** is the set of only those elements that belong to both A and B:

$$\{a, b, c\} \cap \{b, c, d\} = \{b, c\}$$

Remark 3. *Intersection of sets is always a subset of their union:*

$$(A \cap B) \subseteq (A \cup B)$$

Set Membership and IS-A Relation

- Consider the following OAV triplet

  `Dog IS-A mammal`

- The attribute `IS-A` is a relation connecting two concpets, and it can be reprsented by *set membership*:

  $\texttt{Dog} \in \texttt{mammal}$

- Or by *set inclusion*:

  $\texttt{Dog} \subseteq \texttt{mammal}$

Remark 4. *In mathematics, set membership $\in$ and set inclusion $\subseteq$ are very different relations: The former ($\in$) relates an element (object, atom) to a set; the latter ($\subseteq$) relates two sets (e.g. $\varnothing \subseteq \varnothing$ but $\varnothing \notin \varnothing$).*

2 Paradoxes of Naive Set Theory

Paradoxes of Naive Set Theory

- In naive set theory *every* object belongs to some set.
- Is it possible that set $A \in A$?
- Does there exists a set of *all* sets?

Barber
A barber in a village put a note:

I shave everybody, and only those, who do not shave themselves.

Does the barber shave himself?

Russell's paradox

- Bertrand Russell (1872–1970) considered a set R of all sets that do not belong to themselves (i.e. if $A \notin A$ then $A \in R$).
- Then $R \in R$ if and only if $R \notin R$.

Alternative Approaches

Type theory
Initiated by Russell. Objects are of different *types*. Such as

'Hello' is of type *string*, 23 is of type *number*

Axiomatization of Set Theory
Zermelo and Fraenkel introduced restrictions on sets in a form of axioms, including the axiom of Choice, so that the sets are *well-formed*. This is called *ZFC* set theory.

Category theory
Considers objects (e.g. sets) and morphisms (functions) between them.

3 Correspondences and Mappings between Sets

Direct (Cartesian) Product

Definition 13 (Cartesian product)**.** If A and B are two sets, then $A \times B$ is a set with elements (a, b), where $a \in A$ and $b \in B$.

Example 14. Let $A = \{\alpha, \beta, \gamma\}$, $B = \{0, 1\}$. Then

$$A \times B = \{(\alpha, 0), (\beta, 0), (\gamma, 0), (\alpha, 1), (\beta, 1), (\gamma, 1)\}$$

Remark 5. *The order is important:*

$$A \times B \neq B \times A$$

Correspondences and Relations

Definition 15 (Correspondence)**.** • Is any subset $R \subseteq A \times B$.

- The *domain* and *image* of R are:

$$\mathrm{dom}R = \{a \in A : (a, b) \in R\}, \qquad \mathrm{im}R = \{b \in B : (a, b) \in R\}$$

- Inverse correspondence is $R^{-1} = \{(b, a) : \text{if exists } (a, b) \in R\}$

Definition 16 ((Binary) Relation)**.** Is any subset $R \subseteq A \times A$ (i.e. a correspondence between elements of A)

Example 17. Let $R = \{(\alpha, 0), (\gamma, 1)\}$. Then $R^{-1} = \{(0, \alpha), (1, \gamma)\}$ and

$$\mathrm{dom}R = \{\alpha, \gamma\}, \qquad \mathrm{im}R = \{0, 1\}$$

Functions (Mappings)

Definition 18 (Function (mapping))**.** Is a correspondence $f \subset X \times Y$, such that for each $x \in X$ precisely one $y \in Y$ is associated. Denoted

$$f : X \to Y \quad \text{or} \quad y = f(x)$$

Example 19. $y = x^2$ is a function, while $y = \sqrt{x}$ is not.

Remark 6. • *If the inverse $f^{-1} : Y \to X$ is also a function, the f is an isomorphism.*

- *An isomorphism $f : X \to Y$ puts sets X and Y into one-to-one correspondence.*

Classification of Mappings

Definition 20. A mapping $f : X \to Y$ is called

- *Surjective* (or *onto* Y) if $\mathrm{im}f = Y$.
- *Injective* (or *one-one*) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ (i.e. distinct elements have distinct images).
- *Bijective* (or *one-one correspondence*) if f is both surjective and injective.

Remark 7. *A bijective mapping f has inverse f^{-1} and $f^{-1} \circ f(x) = x$.*

Example 21. Linear function $y = \alpha + \beta x$ is bijective, and its inverse is $x = (y - \alpha)/\beta$.