Lecture 2s: Elements of Order Theory

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Binary Relation

Definition 1 (Binary Relation)**.** on set A is any subset $R \subseteq A \times A$.

Notation $(a, b) \in R$ or aRb means 'a is related to b'.

Example 2. Let $A = \{\alpha, \beta, \gamma\}$ be three students, and let α likes β and β likes γ. Then set

$$R = \{(\alpha, \beta), (\beta, \gamma)\}$$

Is a binary relation '*likes*' on A.

Definition 3 (Inverse Relation)**.** Given $R \subseteq A \times A$, its inverse is

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Equality Relation

Definition 4 (Equality $(=, \Delta)$)**.** on set A is

$$\Delta := \{(a, a) : a \in A\}$$

Example 5. Let $A = \{\alpha, \beta, \gamma\}$. Then

$$\Delta = \{(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma)\}$$

- Equality Δ is the diagonal in $A \times A$.
- Equality and its inverse are the same:

$$\Delta = \Delta^{-1}$$

Reflexive Relations

Definition 6. Binary relation $R \subseteq A \times A$ is *reflexive* if aRa for all elements $a \in A$ (i.e. if all elements are related to themselves).

Example 7. Equality Δ is the smallest reflexive binary relation. In fact, all reflexive relations R include equality:

$$\Delta \subseteq R$$

Transitive Relations

Definition 8. Binary relation $R \subseteq A \times A$ is *transitive* if

$$aRb \text{ and } bRc \quad \text{implies} \quad aRc$$

Example 9 (Descedant of)*.* If a is descendant of b and b is a descendant of c, then a is also a descendant of c.

Example 10 (Smaller)*.* Relation 'smaller' $<$ is transitive:

$$1 < 2 \text{ and } 2 < 3 \quad \text{implies} \quad 1 < 3$$

Transitive relations are such that $R \circ R \subseteq R$.

Pre-Order

Definition 11 (Pre-Order)**.** is a binary relation $\lesssim \subseteq A \times A$ that is *reflexive* and *transitive*.

Example 12 (Preference relation)*.* Recall that each pair of elements a, $b \in \Omega$ in a choice set must be comparable. Thus, each element is also comparable with itself, and so preference relation is reflexive. Preference relation is transitive by definition.

Symmetric Relations

Definition 13. Binary relation $R \subseteq A \times A$ is *symmetric* if

$$aRb \quad \text{implies} \quad bRa$$

Example 14*.* 'Married' is a symmetric relation. Indeed, if a is married to b, then b is married to a.

Symmetric relations are such that $R^{-1} \subseteq R$.

Equivalence Relation

Definition 15 (Equivalence)**.** is a binary relation $\sim \subseteq A \times A$ that is *reflexive*, *transitive* and *symmetric*.

Example 16 (Being a relative of)*.* In addition to reflexivity and transitivity, if a is relative of b, then b is also a relative of a. Thus, family relation is an equivalence relation.

Definition 17 (Equivalence Class)**.** of element $a \in A$ is a subset $[a] := \{b \in A : a \sim b\}$

Definition 18 (Quotient Set)**.** of set A with equivalence relation $\sim$ is set $A/\sim$ of all equivalence classes.

Anti-Symmetric Relations

Definition 19. Binary relation $R \subseteq A \times A$ is *anti-symmetric* if

$$aRb \quad \text{implies} \quad \text{not } bRa$$

(or aRb and bRa implies $a = b$)

Example 20 (Offsping of)*.* If a is an offspring of b, then b cannot be an offspring of a.

Anti-symmetric relations are such that $R^{-1} \circ R \subseteq \Delta$.

Order Relation

Definition 21 ((Partial) Order)**.** is a binary relation $\leq \subseteq A \times A$ that is *reflexive*, *transitive* and *anti-symmetric*.

Example 22 (Inclusion)*.* The subset relation $A \subseteq B$ (set inclusion) is reflexive and transitive. Also, because $A \subseteq B$ and $B \subseteq A$ means $A \equiv B$, it is also anti-symmetric.

Example 23*.* Relation 'less than' $\leq$ on real numbers $\mathbb{R}$ is an order relation.

Order-Preserving Mappings

- Let P and Q be two sets and let $f : P \to Q$ be a mapping.
- Suppose P and Q are pre-ordered by $\lesssim_P$ and $\lesssim_Q$ respectively.
- If $p_1 \lesssim p_2$ in P, what happens with $f(p_1)$ and $f(p_2)$ in Q?

Definition 24 (Order-Preserving (Monotone, Isotone))**.** is mapping $f : P \to Q$ between $(P, \lesssim_P)$ and $(Q, \lesssim_Q)$ if

$$p_1 \lesssim_P p_2 \quad \text{implies} \quad f(p_1) \lesssim_Q f(p_2)$$

Example 25 (Utility)*.* is numerical function $u : \Omega \to \mathbb{R}$ from choice set $(\Omega, \lesssim)$ into the set of real numbers $(\mathbb{R}, \leq)$ such that

$$\omega_1 \lesssim \omega_2 \quad \text{if and only if} \quad u(\omega_1) \leq u(\omega_2)$$

where $\lesssim$ is a preference relation (i.e. pre-order) on Ω, and $\leq$ is numerical order. The inverse correspondence $u^{-1} : \mathbb{R} \to \Omega$ is also order-preserving (because of 'iff').

Example 26 (Cost)*.* is $c : \Omega \to \mathbb{R}$ such that

$$\omega_1 \lesssim \omega_2 \quad \text{if and only if} \quad c(\omega_1) \geq c(\omega_2)$$

Thus, cost is *order-reversing* (*antitone*).