Lecture 1s: Elements of Set Theory

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BIS3226

Contents

The bishop gave monkey the banana.
only

Only the bishop gave the monkey the banana.
The *only* bishop gave the monkey the banana.
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The bishop gave the monkey the *only* banana.
The bishop gave the monkey the banana *only*.

1 Sets and Operations on Sets

Sets

Definition 1. A *set* is a collection of objects of some kind with a common property that, given an object and a set, it is possible to decide if the object belongs to the set.

- We denote elements of a set by lower case letters (e.g. $x, y, z, \dots$) and sets by upper case (e.g. $X, Y, Z, \dots$).
- $x \in X$ means 'x belongs to set X'

Example 2. Let $A = \{\text{apple}, \text{orange}, \text{lemon}\}$. Then 'orange' $\in A$ and 'cucumber' $\notin A$.

Example 3. Let $X = \{x_1, \dots, x_{10}\}$. Then $x_2 \in X$ and $x_{100} \notin X$.

How to Write Sets

- Enumerating elements of the set:
$$X = \{a, b, c\}, \qquad Y = \{1, 3, 5, 7, 9\}$$
- Set comprehension
$$X = \{x : P(x)\}$$
where : means '*such that*' and $P(x)$ is a rule that describes the common property.

Example 4. Set Y can be written

$$Y = \{y \in \mathbb{N} : x < 10 \text{ and } x \text{ is odd}\}$$

Example 5.

$$\begin{aligned} P &= \{p : p \text{ is a prime number}\} \\ Y &= \{y : y = x^2, x \in \mathbb{R}\} \end{aligned}$$

Subsets

If every element of set A is also an element of set B, then we write $A \subseteq B$ or $A \subset B$.

$\subset$ subset (proper)
$$\{a, b, c\} \subset \{a, b, c, d\}$$

$\subseteq$ subset or equal
$$\{a, b, c\} \subseteq \{a, b, c\}$$

$\equiv$ equivalence of sets

$$A \equiv B \quad \text{is} \quad A \subseteq B \text{ and } A \supseteq B$$

Remark 1.
- *Empty set $\emptyset = \{\}$ is a subset of any other set $\emptyset \subseteq A$.*
- *Any set is a subset of itself $A \subseteq A$.*

Set Unions and Intersections

Let A and B be two sets, then $A \cup B$ is their union, $A \cap B$ is their intersection.

$\cup$ the result has all elements of both sets (belong to A or B)

$$\{a, b, c\} \cup \{b, c, d\} = \{a, b, c, d\}$$

$\cap$ the result has only the elements that are in both sets (belong to A and B)

$$\{a, b, c\} \cap \{b, c, d\} = \{b, c\}$$

Remark 2. *Intersection of sets is always a subset of their union:*

$$(A \cap B) \subseteq (A \cup B)$$

Special Sets

L. Kronecker:

God gave us the integers; the rest is the work of Man

$\emptyset = \{\}$ an empty set

$\mathbb{N} = \{1, 2, \dots\}$ set of natural numbers

$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ set of integer numbers

$\mathbb{Q} = \{0, 1, -\frac{1}{2}, \frac{20}{7}, \dots\}$ set of rational numbers

$\mathbb{R} = \{0, -1, 3.1, \pi, e, \dots\}$ set of real numbers

$\mathbb{C} = \{1 + i1, -1 + i3.4, \dots\}$ set of complex numbers

Remark 3. *All above sets apart from $\emptyset$ have infinite number of elements.*

Properly Formed Sets

Russell's paradox (after Bertrand Russell, 1872–1970)

A barber in a village put a note:

'I shave everybody who does not shave themselves'

Question 1. *Does the barber shave himself?*

The following sentence is false The previous sentence is true

2 Correspondences and Mappings between Sets

Direct (Cartesian) Product

Definition 6 (Cartesian product)**.** If A and B are two sets, then $A \times B$ is a set with elements (a, b), where $a \in A$ and $b \in B$.

Example 7. Let $A = \{\alpha, \beta, \gamma\}$, $B = \{0, 1\}$. Then

$$A \times B = \{(\alpha, 0), (\beta, 0), (\gamma, 0), (\alpha, 1), (\beta, 1), (\gamma, 1)\}$$

Remark 4. *The order is important:*

$$A \times B \neq B \times A$$

Correspondences

Definition 8 (Correspondence)**.** Is any subset $R \subseteq A \times B$. The *domain* and *image* (range) of R are:

$$\mathrm{dom}R = \{a \in A : (a, b) \in R\}, \qquad \mathrm{im}R = \{b \in B : (a, b) \in R\}$$

Inverse correspondence is $R^{-1} = \{(b, a) : (a, b) \in R\}$

Example 9. Let $R = \{(\alpha, 0), (\gamma, 1)\}$. Then

$$\mathrm{dom}R = \{\alpha, \gamma\}, \qquad \mathrm{im}R = \{0, 1\}$$

and $R^{-1} = \{(0, \alpha), (1, \gamma)\}$

Functions (Mappings)

Definition 10 (Function (mapping))**.** A function (mapping) from X into Y

$$f : X \to Y\,, \quad \text{or} \quad y = f(x)$$

is a correspondence $f \subset X \times Y$, such that $\text{dom}R = X$, and for each $x \in X$ precisely one $y \in Y$ is associated.

Example 11. $y = x^2$ is a function, while $y = \sqrt{x}$ is not.

Classification of Mappings

Definition 12. A mapping $f : X \to Y$ is called

- *Surjective* (or *onto* Y) if $\text{im}f = Y$.
- *Injective* (or *one–one*) if $f(x_1) = f(x_2)$ only if $x_1 = x_2$ (i.e. distinct elements have distinct images).
- *Bijective* (or *one–one correspondence*) if f is both surjective and injective.

If a mapping is bijective, then there exists an inverse mapping

$$f^{-1} : Y \to X$$

Example 13. Linear function $y = \alpha + \beta x$ is bijective, and its inverse is $x = \beta^{-1}(y - \alpha)$.