Dynamics of Information and Optimal Control of Reproduction in Evolutionary Systems

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Introduction

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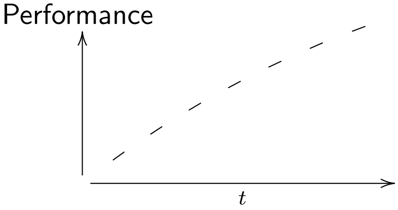

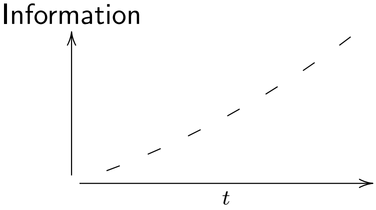

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- *Y is a module over $X \subset Y$ with* $\|y\|_1 = |\langle 1, y\rangle|$

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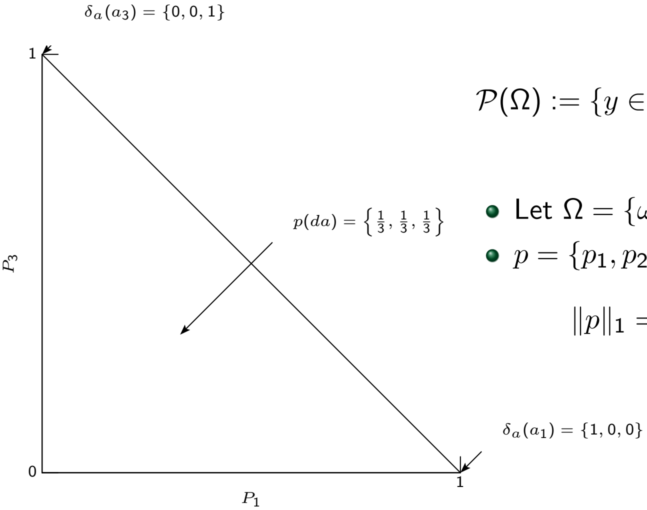

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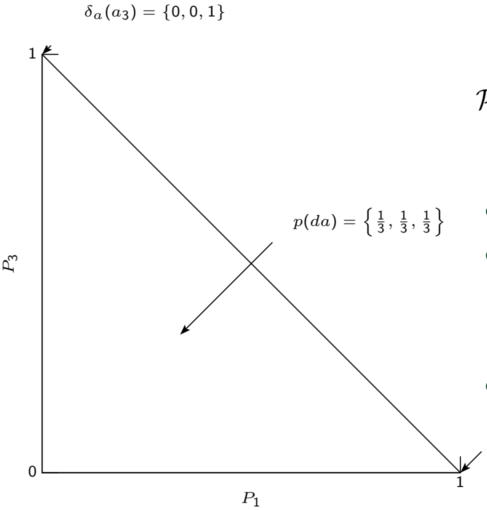

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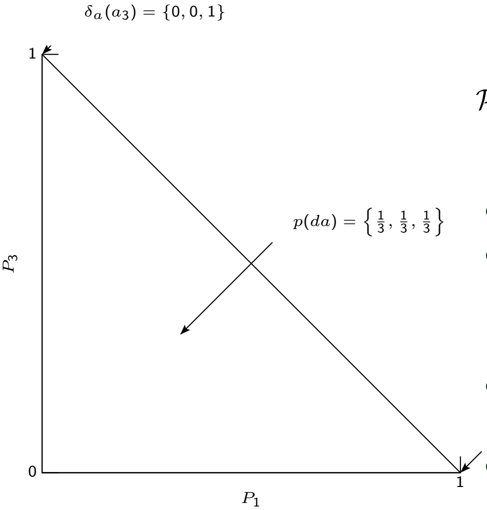

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- Choquet simplex if Ω is infinite.

Measures of Performance and Information

Expected utility

Let $(\Omega, \mathcal{R}, p)$ be a probability space, and $x : \Omega \to \mathbb{R}$ a utility. Then

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A closed functional $F : Y \to \mathbb{R} \cup \{\infty\}$ with $F(p)|_{\mathcal{P}} = I(p, q)$, where $I : \mathcal{P} \times \mathcal{P} \to \mathbb{R}_+ \cup \{\infty\}$ is an information distance. For example

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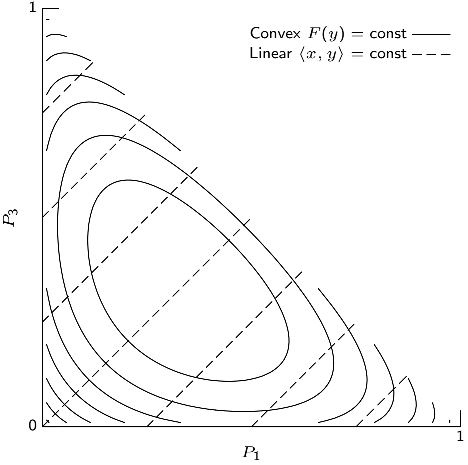
Remark

Other information distances are

$$I_V(p, q) = \|p - q\||_1 \,, \quad I_F(p, q) = 2 \arccos\langle 1, p^{1/2} q^{1/2} \rangle$$

Information Dynamics



- $\mathbb{E}_p\{x\} = \langle x, p\rangle$ is linear.

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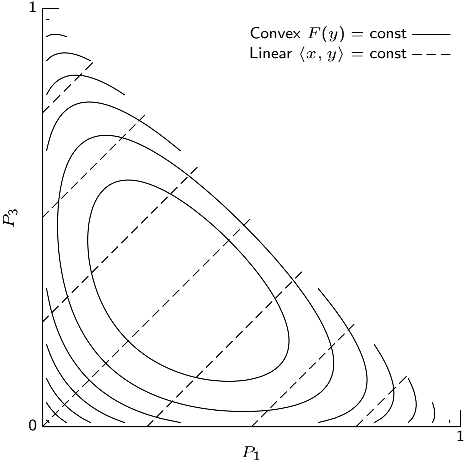

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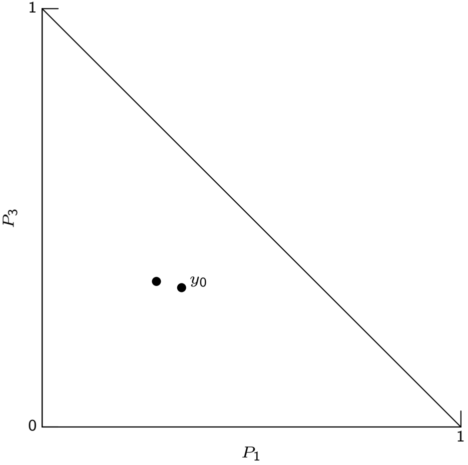

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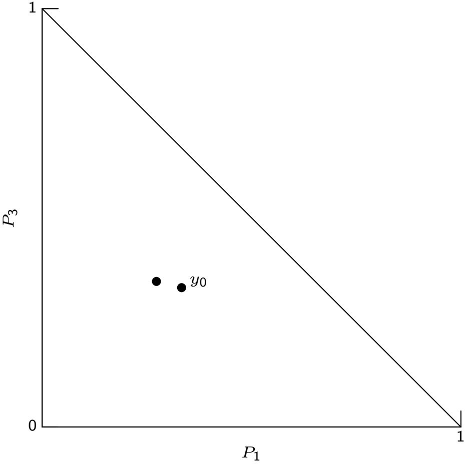

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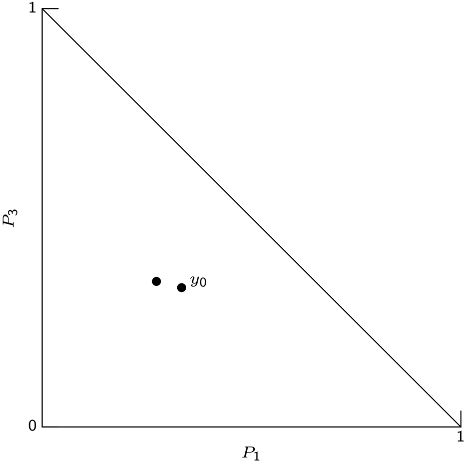

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- Integral solution

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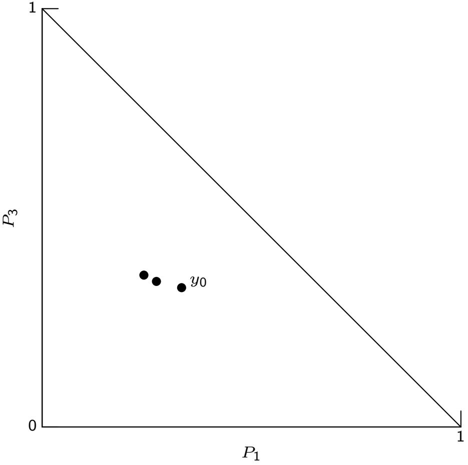

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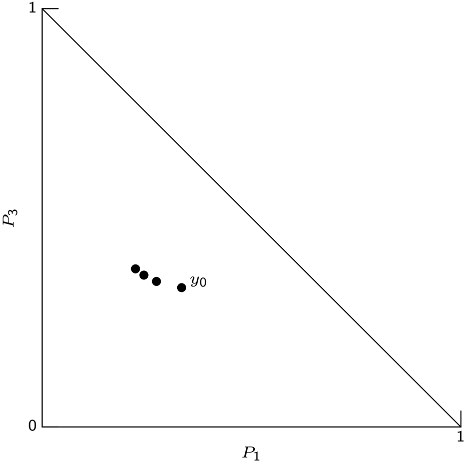

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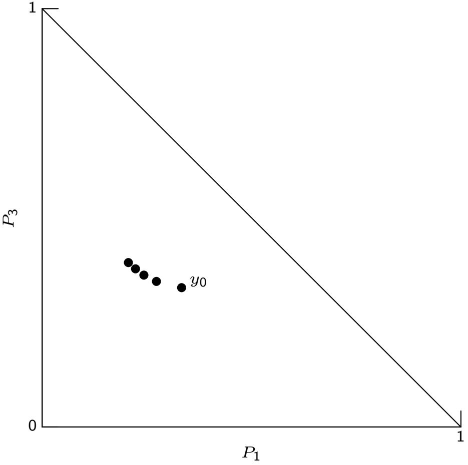

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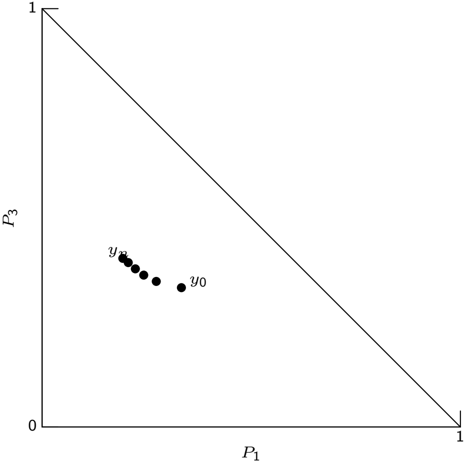

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Question

Which transformations (controls) of the system lead to $U_x(I)$ on $\mathcal{P}(\Omega)$?

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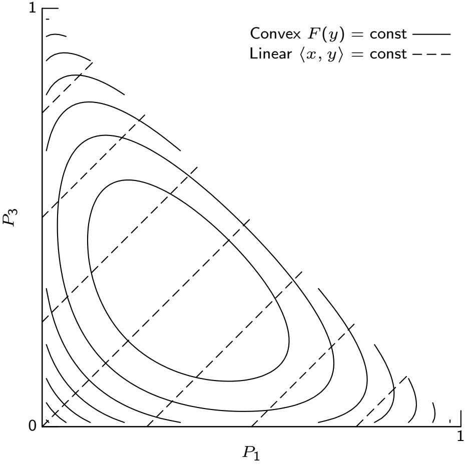

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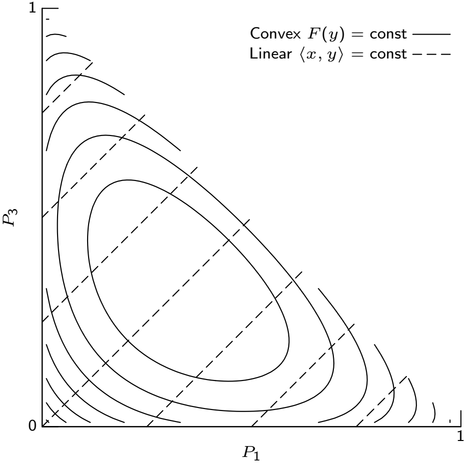

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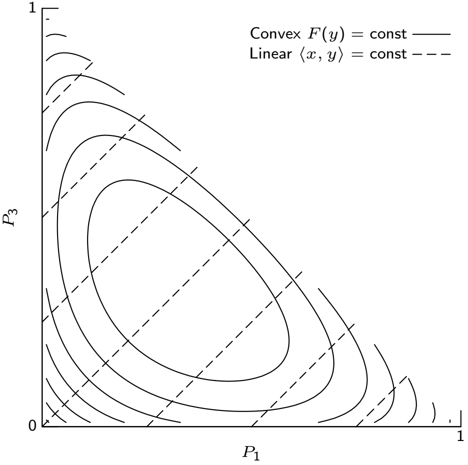

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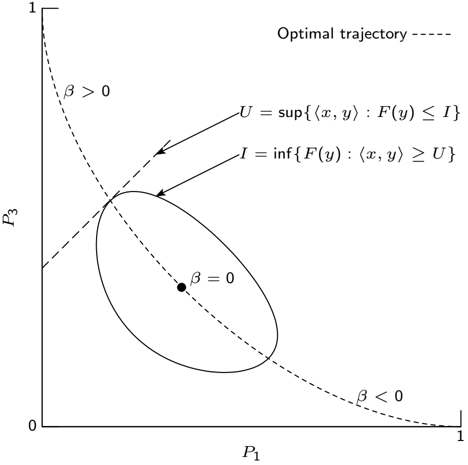

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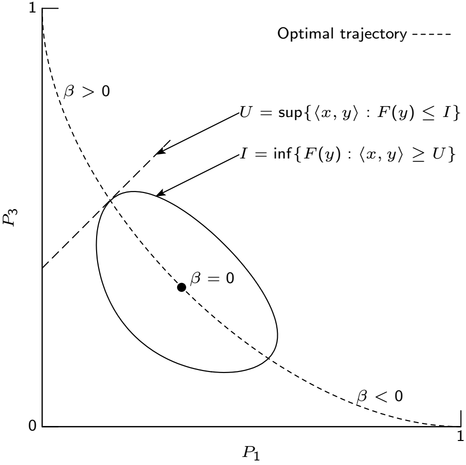

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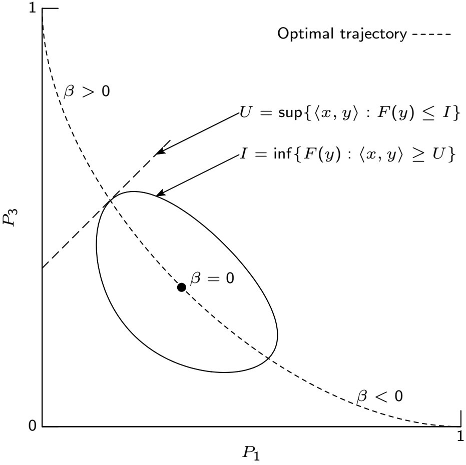

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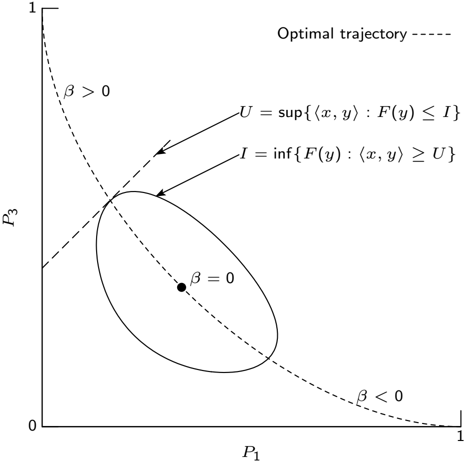

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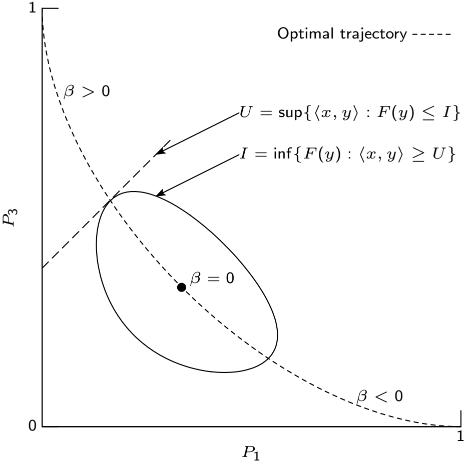

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Question

Time or Information?

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Evolution as an Information Dynamic System

- EPSRC Sandpit '*Math of Life*' (July, 2009):

- Three year project

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- Total value $\pounds 519,648$

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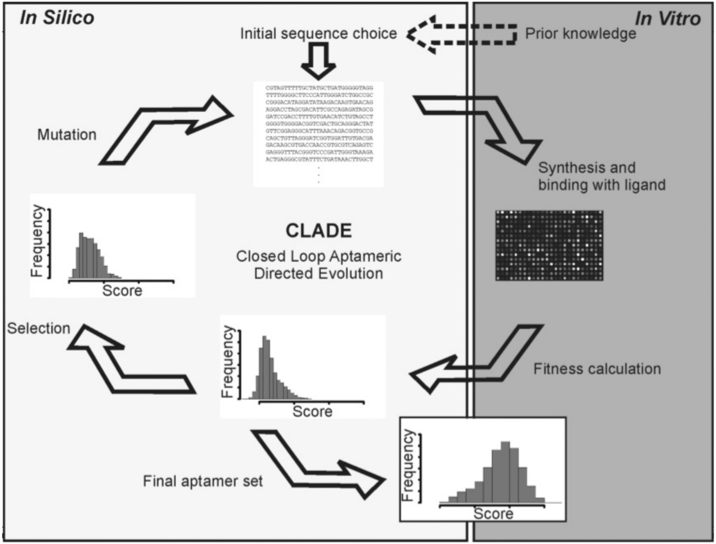
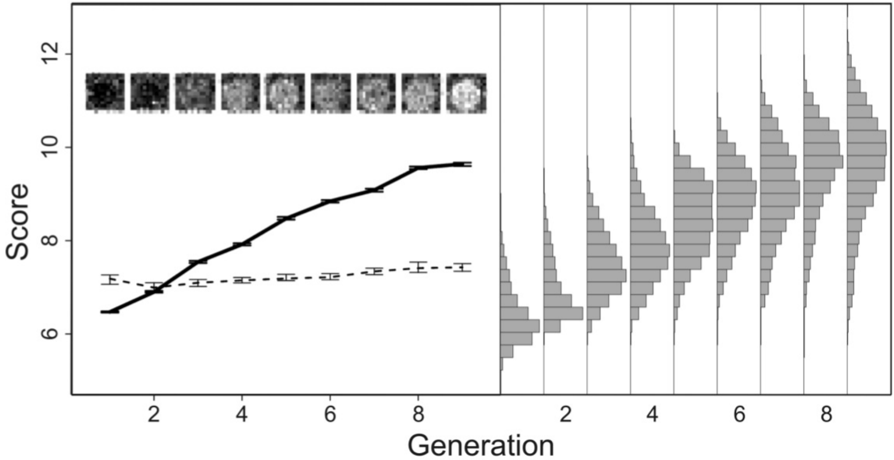
To use the theory for predictions and models, experiments for its verification, as well as inform and extend the theory

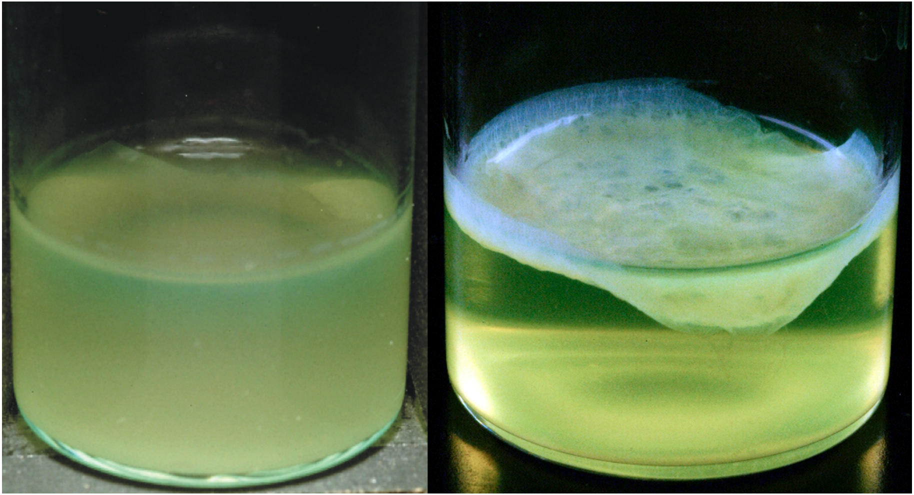
In Vitro, In Silico (C. Knight (2009))



Evolution *In Vitro*



Evolution *In Vivo*



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Definition (Monotonic landscape)

f is locally monotonic (isomorphic) relative to a metric d, if there exist $S(\top, l) := \{\omega : d(\top, \omega) < l\}$, $\top = \vee\Omega$ such that $\forall\, a, b \in S(\top, l)$:

$$-d(\top, a) \leq -d(\top, b) \quad \Longrightarrow (\iff) \quad f(a) \leq f(b)$$

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Question

How to control parameters to achieve optimal evolution?

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- where for $\alpha = 2$ (binary strings):

$$|S(a,r) \cap S(\top,k)|_l = \begin{cases} 0\,, & \text{if } l_+ \notin \mathbb{N} \cup \{0\} \\ \binom{n-l}{l_+}\binom{l}{l_-}\,, & \text{otherwise} \end{cases}$$

with $l_+ = \frac{1}{2}(r + k - l)$ and $l_- = \frac{1}{2}(l - |r-k|) = \min\{r,k\} - l_+$.

Analysis Using Absorbing Markov Chain

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$$p_{t+1} = T_\theta p_t = \sum_{l=0}^{n} P_\theta(k \mid l) P(l) \quad \text{or} \quad p_{t+m} = T_\theta^m p_t$$

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- Given $p_0 = P(l)$, compute the expected time of convergence:

$$\mathbb{E}_P\{t\} = \sum_{l=1}^{n} t_l P(l) = \sum_{l=1}^{n} P(l) \sum_{k=1}^{n} n_{lk}$$

Optimisation of Mutation in One Step

- The expected fitness after transformation $a \mapsto b$ is

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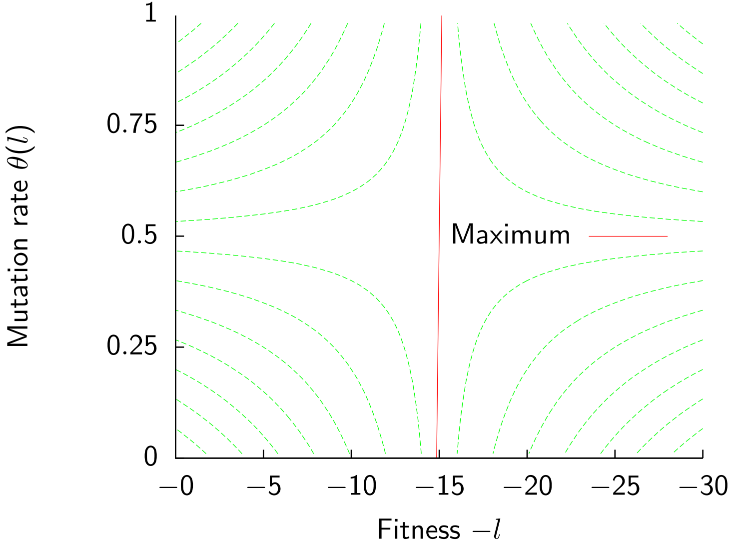
- If $f(b) = -d(\top, b)$ and $P(a) = \delta_a(\omega)$, then

$$\bar{\theta}(l) := \begin{cases} 0 & \text{if } l < n/2 \\ 1 & \text{if } l > n/2 \end{cases}$$

Step function



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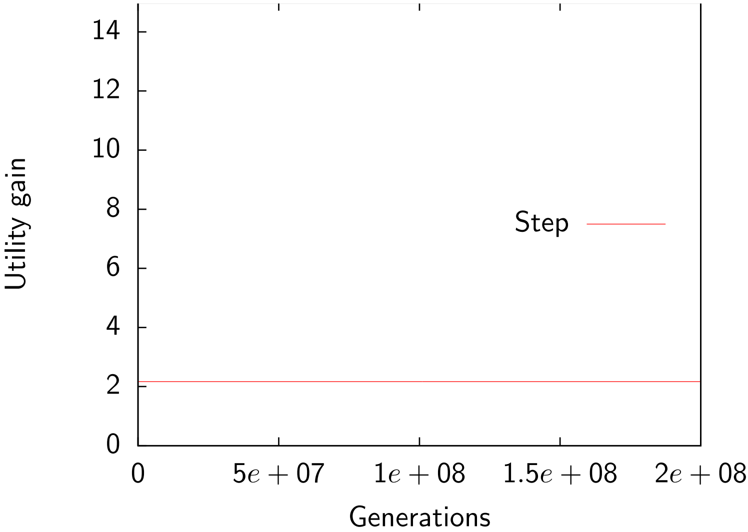

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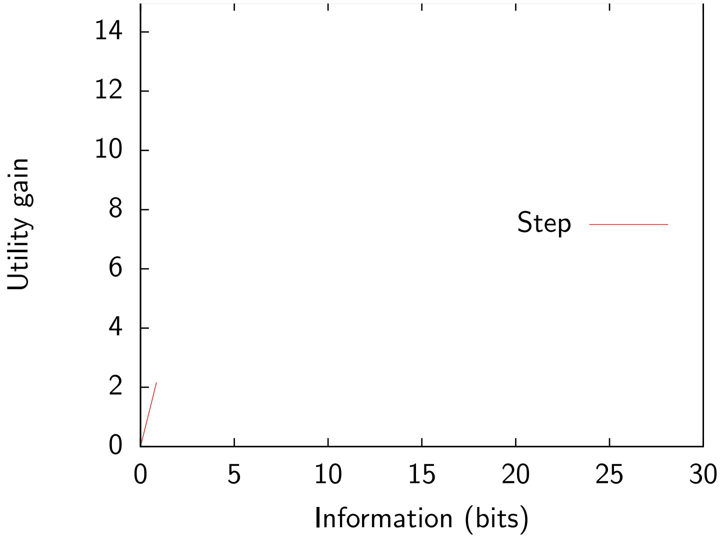

Maximising Fitness Growth

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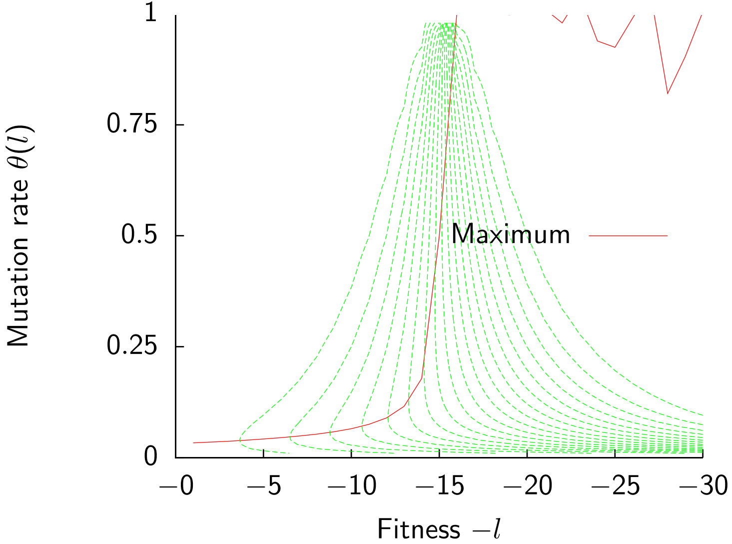
- The maximiser of $u(\theta)$ can be found numerically.

$\Delta\mathbb{E}\{f\}$ function



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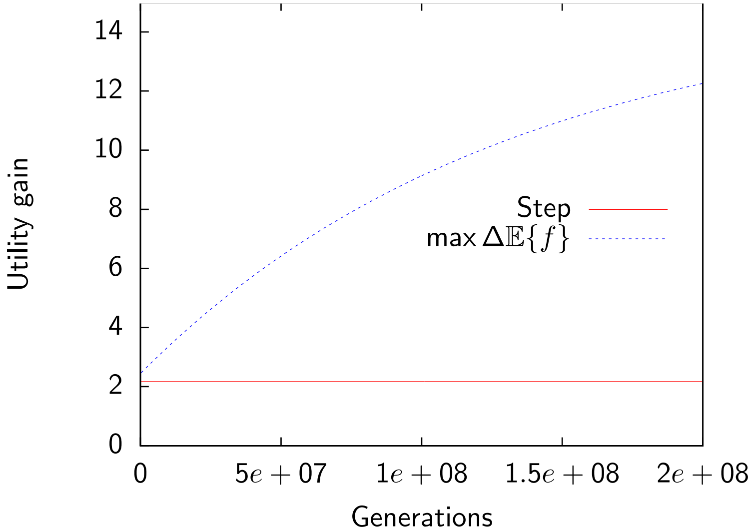

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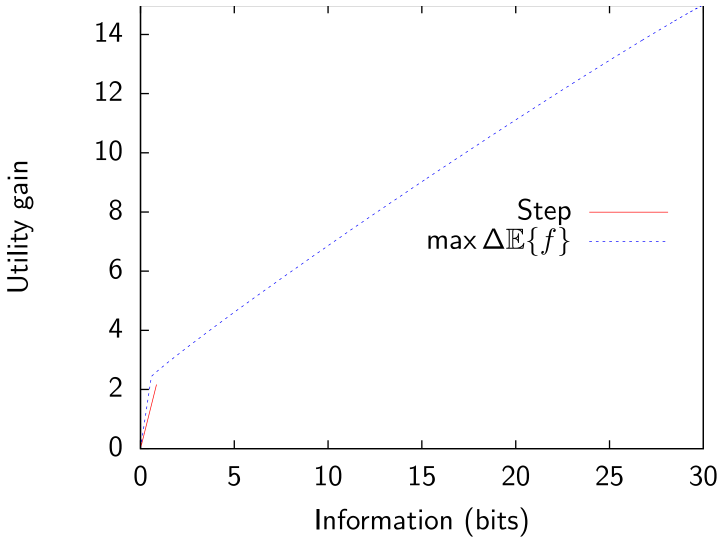

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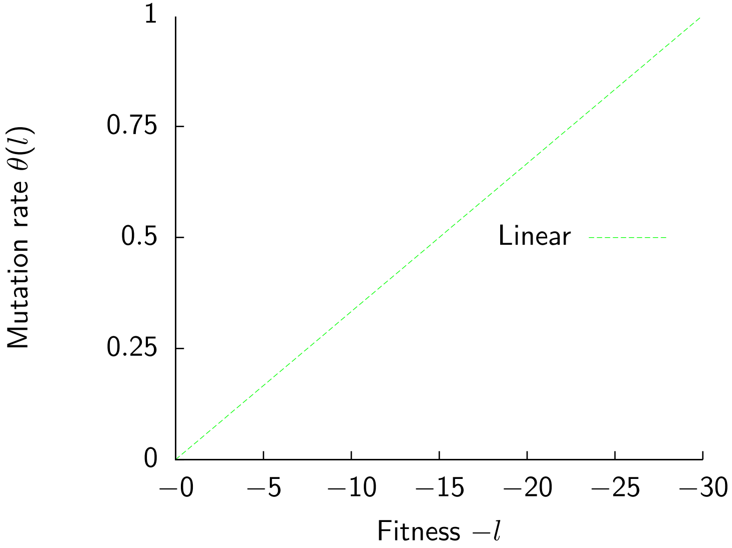
- Condition $dP_\theta(0 \mid l \neq 0)/d\theta = 0$ is satisfied for $l = n\theta$, which gives linear function:
$$\theta(l) = \frac{l}{n}$$

Linear function



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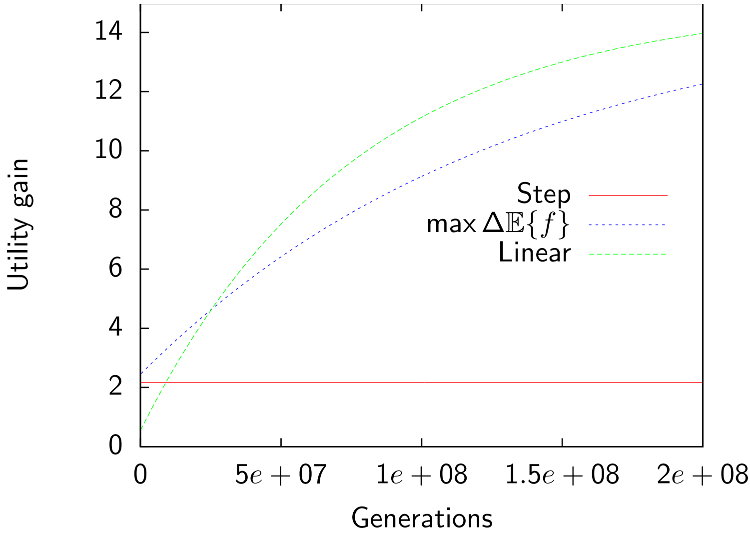

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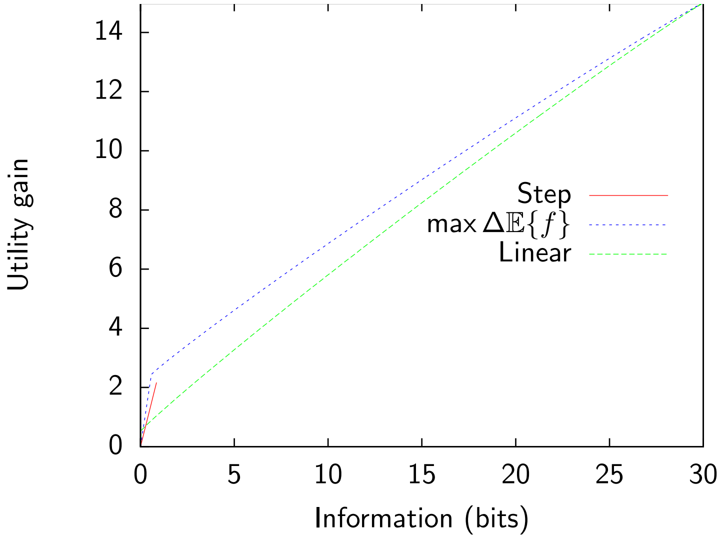

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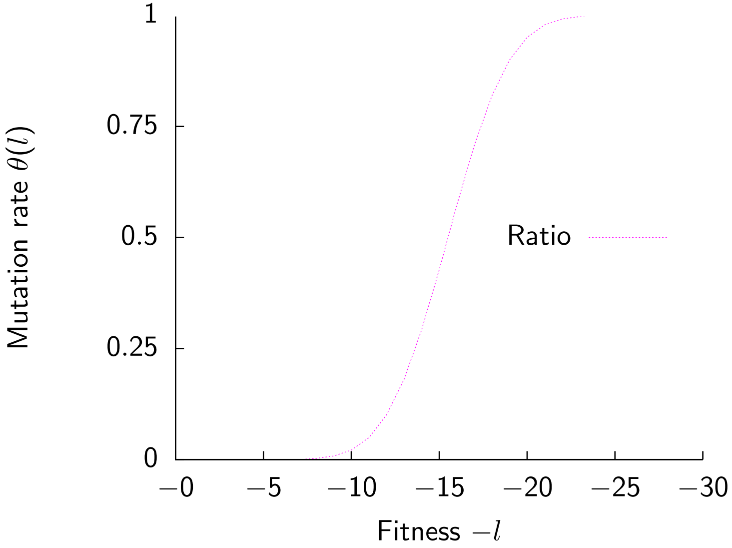
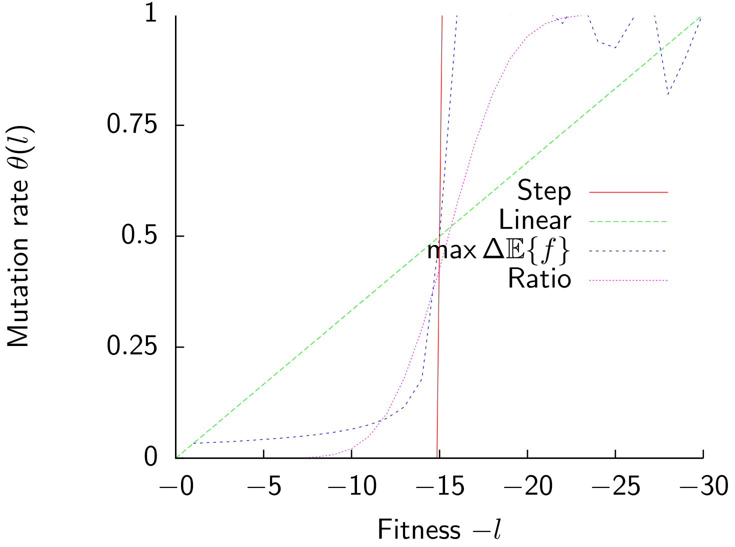
- The probability of 'improvement' is given by

$$P(k < l) = \sum_{k=0}^{l-1} P(k) = 2^{-n} \sum_{k=0}^{l-1} \binom{n}{k}$$

‘Informed’ Mutation function



Comparing Mutation Rate Functions



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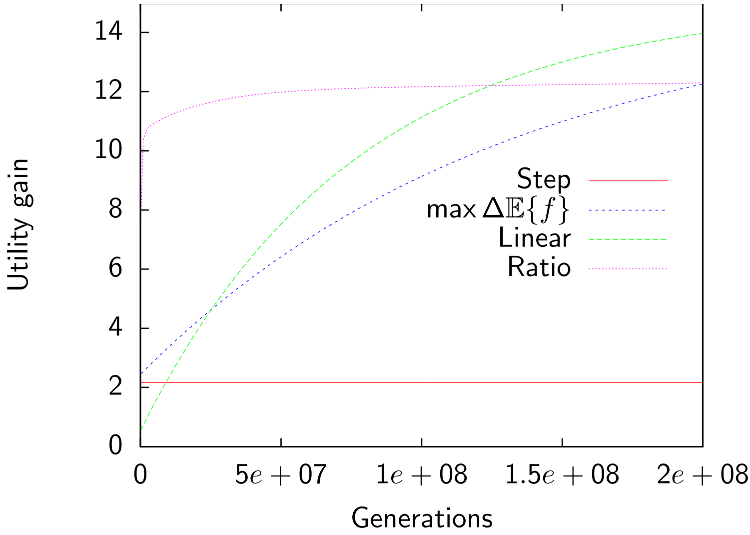

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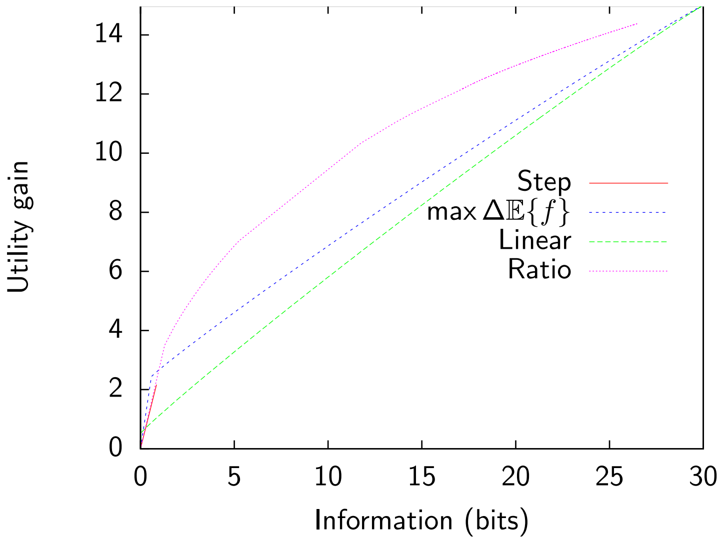
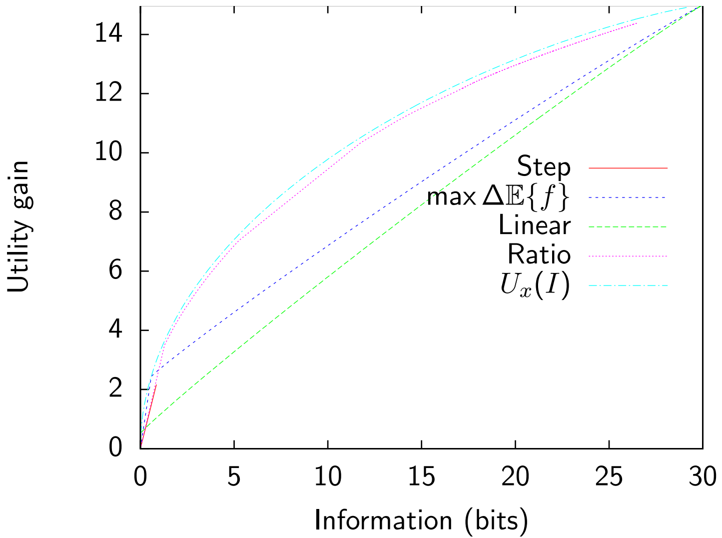


Theoretical $U_x(I)$



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- There is a problem of relating transformations of the system to information dynamics.

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