Parameterized Uncertain Reasoning Approach Based on a Lattice-Valued Logic

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Abstract. This paper presents a parameterized reasoning approach with uncertainty based on a lattice-valued logic system. In this uncertain reasoning approach, some parameters are used to represent uncertainty arising from different sources, which is a common phenomenon in rule-based systems. In our system, reasoning with different parameter values means reasoning with different levels of belief and consistency. Some methods are presented for selecting appropriate parameter values during the uncertain reasoning process which allow us to find suitable parameter values to meet the diverse practical and theoretical requirements.

Keywords: Parameterized reasoning, uncertainty, lattice-valued logic, rule-based systems

1 Introduction

Rules are one of the most common forms for representing knowledge. Rule-based systems (or knowledge-based systems) using IF-THEN rules to represent knowledge and to reason with it, have been applied successfully in many areas [10]. A crucial issue in rule-based systems is to utilize all information available to analyze the current situation as expressed by the rules, and infer the consequences which will lead to corresponding actions. Often, this is a process of uncertain reasoning, *i.e.*, inferring conclusions based on rules and new information under uncertainty.

Uncertainty may arise from different aspects of the reality that is being represented. For example, suppose that we are evaluating the quality of a car from 4 aspects: price, safety, comfort and fuel economy, which will be discussed in more detail as an illustrative example in the paper. Uncertainty may arise from subjective judgement about a car, *e.g.*, "this car is quite safe", where "quite" depicts the truth degree of the judgement or evaluation about the safety of the car. There is also uncertainty on the belief degree of the experts on the rule, *e.g.*, "the rule is highly true". Uncertainty may also exist in the reasoning process from the observations of a car to the overall evaluation due to the subjective and ambiguous situations. And there are sometimes contradicting observations or opinions about a car, which is represented by the consistency level of the observations. We use different parameters in the uncertain reasoning approach to represent all these different types of sources of uncertainty, which vary from problem to problem.

From the viewpoint of symbolism, the confidence and rationality of uncertain reasoning relies on logics which are extensions of classical logic, so-called non-classical logics [11]. Zadeh [19] developed a theory of uncertain reasoning based on the notion of linguistic variable and fuzzy logic, which then influenced research of uncertain reasoning with strict logical foundation. Pavelka [9] and Novak [8] then laid the foundation for the research of uncertain reasoning theory and methods based on strict logic system. Many researchers have made many important progress in this area [4], [5], [6], [15]. There are also some works related to reasoning method with uncertainty from different sources, such as, Larsen and Yager [7] presented a method for crisis recognition under uncertainty in the framework of possibility logic by using belief measure to reflect the type of uncertainty in the observations and knowledge base. Benferhat and Sossai [1] proposed a method for reasoning with multiple-source information by merging uncertain knowledge bases, provided by different sources, into a new possibilistic knowledge base in the framework of possibilistic logic. Zhou *et.al.* [20] gave a graded reasoning method in the framework of n-valued R_0-logic $\mathcal{L}_n^*$. Sottara *et.al.* [12] introduced an architecture depending on a number of configuration parameters which could be set by the user, individually or as a whole for the entire rule base.

In this paper, we will propose a parameterized uncertain reasoning method which will take the advantage of direct reasoning with observed information to get the result, without the underlying numerical approximation needed by fuzzy set based method. This uncertain reasoning method is based on lattice-valued logic with truth values in lattice implication algebra (LIA) [15], which is a type of non-classical logic.

The paper is organized as follows. First some related concepts and results about lattice-valued logic and lattice implication algebra are recalled and revised. Then, review and analysis of the uncertain reasoning approach based on lattice-valued logic is given, followed by the introduction of methods for parameter selection when applying the uncertain reasoning approach in a specified lattice-valued logic system, $\mathscr{L}_{2nf}$. Finally, an example is given to illustrate the proposed method.

2 Lattice-Valued First-Order Logic

Lattice implication algebra [15] is a kind of lattice-valued logical algebra, which is the truth-value field of lattice-valued logic. It has been shown in [15], [18] that lattice implication algebra defines a residuated lattice [9], which possesses the common features in various fuzzy logical systems based on the different particular algebraic structures [13].

In the following, we denote L as a lattice implication algebra (LIA) and L_{vfl} as the lattice-valued first-order logic based on L. The generalized quantifiers in L_{vfl} is denoted as $\mathbf{Q}_u$, where $u \in U$, U is an index set, which can be seen as a generalization of $\forall$ and $\exists$. The set of all well-formed formulae (wffs), such as $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \to \psi$, $(\mathbf{Q}_u x)\varphi$, in L_{vfl} is denoted as $\mathcal{F}_f$. A well-formed formula is called a formula for short. In the car evaluation example, formulas φ, ψ will be used to represent the attibutes of cars. For example, $\varphi_i(x)$ represents the i-th attribute, say comfort of car x. Let φ, $\psi \in \mathcal{F}_f$, we also denote

$$\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi) \tag{1}$$

$$\varphi \otimes \psi = (\varphi \to \psi')' \tag{2}$$

An interpretation of wffs in L_{vfl} is a mapping $\mathscr{D}_{\mathcal{F}_f} : \mathcal{F}_f \longrightarrow L$, which is to assign wffs truth degrees, *e.g.*, assign truth degrees to the attributes of cars in the car evaluation problem. The set of interpretations of wffs is denoted as

$$\mathcal{I}_h \subseteq \mathcal{I}_H \triangleq \{\mathscr{D}_{\mathscr{F}_f} | \mathscr{D}_{\mathscr{F}_f} \text{is an interpretation of wffs}\}.$$

In the following, we also call $\mathcal{I} \subseteq \mathscr{F}_L(\mathcal{F}_f)$ as the set of interpretations of wffs, where $\mathscr{F}_L(\mathcal{F}_f)$ is the set of all L-type fuzzy subsets on $\mathcal{F}_f$.

Definition 1 [15] *Let $D_n \subseteq \mathscr{F}_f^n$. A mapping $r_n : D_n \longrightarrow \mathscr{F}_f$ is called an n-ary partial operation of $\mathscr{F}_f$, where D_n is the domain of r_n, also denoted by $D_n(r_n)$.*

Definition 2 [15] *A mapping $t_n : L^n \longrightarrow L$ is said to be an n-ary truth-valued operation on L, if*
(1). $\alpha \to t_n(\alpha_1, \cdots, \alpha_n) \geq t_n(\alpha \to \alpha_1, \cdots, \alpha \to \alpha_n)$ *holds for any $\alpha \in L$ and $(\alpha_1, \cdots, \alpha_n) \in L^n$.*
(2). *t_n is isotone in each argument.*

We denote

$$R_n \subseteq \{r_n \mid r_n \text{ is an n-ary partial operation of } \mathcal{F}_f\},$$
$$T_n \subseteq \{t_n \mid t_n \text{ is an n-ary truth-valued operation on } L\},$$
$$\mathscr{R}_n \subseteq R_n \times T_n, \quad \mathscr{R} \subseteq \bigcup_{n=0}^{+\infty} \mathscr{R}_n.$$

If $(r,t) \in \mathscr{R}_n$, then (r,t) is called an n-ary rule of inference in L_{vfl}.

It can be seen that there are two parts for an inference rule in L_{vfl}, r is for the formal deduction of formulas, and t is for the transformation of truth values of these formulas.

Definition 3 [15] *Let $X \in \mathscr{F}_L(\mathcal{F}_f)$, $(r,t) \in \mathscr{R}_n$, $\alpha \in L$. If*

$$X \circ r \supseteq \alpha \otimes (t \circ \prod^n X) \tag{3}$$

holds, then X is said to be α-I type closed w.r.t. (r,t). If

$$X \circ r \supseteq t \circ \prod^n (\alpha \otimes X) \tag{4}$$

holds, then X is said to be α-II type closed w.r.t. (r,t), where $\circ$ means the composition of functions, and $\prod$ is cartesian product.

If for any $(r,t) \in \mathscr{R}$, X is α-i type closed w.r.t. (r,t), then X is said to be α-i type closed w.r.t. $\mathscr{R}$, i =I, II.

Definition 4 [15] *Let $\alpha \in L$, $\mathscr{R}$ is said to be α-i type sound w.r.t. $\mathcal{I}$, if T is α-i type closed w.r.t. $\mathscr{R}$ holds for any $T \in \mathcal{I}$, i =I, II.*

Here, α can be thought of as the level of soundness of the inference rule in lattice-valued logic, which can be interpreted as the belief degree of the decision rule in the rule base for car evaluation problem.

Definition 5 [15] *Let X, $Y \in \mathscr{F}_L(\mathcal{F}_f)$, $\varphi \in F_f$, α, $\beta \in L$, i=I, II.*
(1).

$$\begin{aligned} C_{\mathcal{I}} : \mathscr{F}_L(\mathcal{F}_f) &\longrightarrow \mathscr{F}_L(\mathcal{F}_f), \\ X &\longmapsto C_{\mathcal{I}}^X, \end{aligned}$$

$$C_{\mathcal{I}}^X(\varphi) \triangleq \bigwedge_{T\in\mathcal{I}} \Big(\bigwedge_{\varphi\in\mathcal{F}_f} (X(\varphi) \to T(\varphi)) \to T(\varphi)\Big), \tag{5}$$

(2).

$$\begin{aligned} C^{\beta}_{(C_{\mathcal{I}}^{\emptyset},\mathscr{R}(\alpha-i))} : \mathscr{F}_L(\mathcal{F}_f) &\longrightarrow \mathscr{F}_L(\mathcal{F}_f), \\ X &\longmapsto C^{\beta,X}_{(C_{\mathcal{I}}^{\emptyset},\mathscr{R}(\alpha-i))}, \end{aligned}$$

$$\begin{aligned} C^{\beta,X}_{(C_{\mathcal{I}}^{\emptyset},\mathscr{R}(\alpha-i))}(\varphi) \triangleq \bigwedge \{\, & Y(\varphi) \mid Y \supseteq \beta \otimes (C_{\mathcal{I}}^{\emptyset} \cup X), \\ & Y \text{ is } \alpha\text{-}i \text{ type closed w.r.t. } \mathscr{R} \,\}. \end{aligned} \tag{6}$$

$C_{\mathcal{I}}$ is a semantic closure operator reflecting the transformation of truth values from X to $C_{\mathcal{I}}^X$ under interpretation set $\mathcal{I}$, which will be used to get the uncertain reasoning consequence. In the car evaluation problem, $C_{\mathcal{I}}^X$ gives the degree to which the evaluation X of a specified car can be included in or can reflect a general evaluation $\mathcal{I}$ of cars. β means the degree to which can we get the evaluation result from the observations of a car and established rules.

Definition 6 [15] *Let $X \in \mathscr{F}_L(\mathcal{F}_f)$, $\varphi \in \mathcal{F}_f$, θ, α, $\beta \in L$. $(P^i, (n), X, (\varphi,\theta) - (\alpha,\beta))$ is said to be an (α,β)-i type proof of φ from X with the truth-valued degree θ (shortly, θ-(α,β)-i type proof of φ from X), if the mapping*

$$\begin{aligned} P^i : (n) &\longrightarrow \mathcal{F}_f \times L,\ (n) = \{1, 2, \cdots, n\} \\ j &\longmapsto (\varphi_j, \theta_j), \end{aligned}$$

satisfies:
(1). $(\varphi_n, \theta_n) = (\varphi, \theta)$ *and*
(2). $\theta_j = \beta \otimes C_{\mathcal{I}}^{\emptyset}(\varphi_j)$, *or*
(3). $\theta_j = \beta \otimes X(\varphi_j)$, *or*
(4). *there exist* $j_1, \cdots, j_k < j$, *and* $(r,t) \in \mathscr{R}_k$, *such that*

$$\begin{aligned} (\varphi_j, \theta_j) &= (r(\varphi_{j_1}, \cdots, \varphi_{j_k}), \alpha \otimes t(\theta_{j_1}, \cdots, \theta_{j_k})),\ i = I, \\ (\varphi_j, \theta_j) &= (r(\varphi_{j_1}, \cdots, \varphi_{j_k}), t(\alpha \otimes \theta_{j_1}, \cdots, \alpha \otimes \theta_{j_k})),\ i = II, \end{aligned}$$

where n is said to be the length of θ-(α,β)-i type proof of φ from X under P^i, and is denoted by $l(P^i)$, i =I, II.

Definition 7 [15] *Let* $X \in \mathscr{F}_L(\mathcal{F}_f)$*,* $\tau \in L$*, i=I, II. If*

$$\bigvee\{C^{\beta,X}_{(C^{\emptyset}_{\mathcal{I}},\mathscr{R}(\alpha-i))}(\varphi)\otimes C^{\beta,X}_{(C^{\emptyset}_{\mathcal{I}},\mathscr{R}(\alpha-i))}(\varphi')|\varphi \in F_f\} \leq \tau, \tag{7}$$

then X *is said to be* τ'*-i type consistent w.r.t.* $(\alpha, \beta, \mathcal{I})$*.*

τ' represents the level of consistency of X which can be antecedent or consequent in the inference rule. For example, there may be some conflicting observations of a car or conflicting rules in the rule-base, and τ' is used to represent the degree to which they are not conflicting, *i.e.*, consistent.

Theorem 8 [15] *Let* $X \in \mathscr{F}_L(\mathcal{F}_f)$*,* α*,* $\beta \in L$*, and the truth-valued operations in* $\mathscr{R}$ *satisfy the finite semicontinuity. Then for any* $\varphi \in F_f$*, i =I, II,*

$$C^{\beta,X}_{(C^{\emptyset}_{\mathcal{I}},\mathscr{R}(\alpha-i))}(\varphi)=\bigvee\{\,\theta \mid \exists(P^i,(n),X,(\varphi,\theta)-(\alpha,\beta))\}, \tag{8}$$

where $(P^i,(n),X,(\varphi,\theta)-(\alpha,\beta))$ *is an* (α,β)*-i type proof of* φ *from* X *with the truth-valued degree* θ*.*

Theorem 9 [15] *Let* α*,* $\beta \in L$*, and for any* $X \in \mathscr{F}_L(\mathcal{F}_f)$*,* $\mathscr{R}$ *is* α*-i type sound w.r.t.* $\mathcal{I}$*, and* $C^{\beta,X}_{(C^{\emptyset}_{\mathcal{I}},\mathscr{R}(\alpha-i))} \in \mathcal{I}$*. Then for i =I, II,*

$$C^{\beta,X}_{(C^{\emptyset}_{\mathcal{I}},\mathscr{R}(\alpha-i))} = C^{\beta\otimes X}_{\mathcal{I}}. \tag{9}$$

Theorems 8 and 9 state the soundness and completeness of lattice-valued logic to some degree, *i.e.*, the compatibility between syntax and semantics in lattice-valued logic.

3 Uncertain Reasoning Approach Based on Lattice-Valued Logic L_{vfl}

We take the typical uncertain reasoning model to explain the uncertain reasoning approach based on lattice-valued logic L_{vfl}. It should be noticed that this model is not only a single-input single-output model, because X and Y are actually assignments of truth degrees to a set of formulas (attibutes).

$$\begin{array}{rl} \textit{Rule}: & \text{If } X, \text{ then } Y, \\ \textit{Fact}: & \widetilde{X}, \\ \hline \textit{Conclusion}: & \widetilde{Y}, \end{array} \tag{10}$$

where X, Y, $\widetilde{X}$, $\widetilde{Y} \in \mathscr{F}_L(\mathcal{F}_f)$.

Based on the above model, an uncertain reasoning theory and approach has been proposed in [2], which has strict logic foundation, *i.e.*, lattice-valued first-order logic L_{vfl}. The uncertain reasoning consequence is expressed as:

$$\widetilde{Y} = C^{\beta\otimes\widetilde{X}}_{\mathcal{I}}, \tag{11}$$

where $C_{\mathcal{I}}$ is defined in Definition 5. Here, we need the uncertain reasoning model (10) to be $(\alpha, \beta, \tau, \mathcal{I})$-$i$ type regular [14], [2], *i.e.*, there exist α, β, $\tau \in L$, $\mathcal{I} \subseteq \mathscr{F}_L(\mathcal{F}_f)$ and $\mathscr{R}$ such that X, Y, $\widetilde{X}$ is τ'-i type consistent w.r.t. $(\alpha, \beta, \mathcal{I})$, and $C_{\mathcal{I}}^X \supseteq \tau' \otimes Y$.

Furthermore, if the above selected α, β, τ, $\mathcal{I}$ and $\mathscr{R}$ make $C^{\beta,\tilde{X}}_{(C^{\emptyset}_{\mathcal{I}}, \mathscr{R}(\alpha-i))} \in \mathcal{I}$, then from Theorems 8 and 9, the uncertain reasoning consequence can also be obtained by a strict formal deduction in L_{vfl}, *i.e.*, the uncertain reasoning consequence is not only semantically sound, but also syntactically provable to some degree.

It should be noticed that the above conditions for parameters are always satisfiable. For example, equation (7) always holds for $\tau = I$, *i.e.*, any X is consistent at O level, this of course is useless. So, what we need to do is to choose reasonable values, according to practical and logical requirements, for these parameters under certain situations.

In [2], we have chosen a set of inference rules $\mathscr{R}^*$, including three special rules and five classes of rules, which can cover rules used frequently in most cases.

$$\begin{aligned}\mathscr{R}^* = &\{(r_2^0, t_2^*), (r_2^*, t_2^*), (r_2^{\triangle}, t_2^*)\} \cup \{(r_1^{\theta_0}, t_1^{\theta_0}) \,|\, \theta_0 \in L\} \\ &\cup \{(r_1^u, t_1) \mid u \in U\} \cup \{(r_2^u, t_1) \mid u \in U\} \\ &\cup \{(r_3^u, t_1) \mid u \in U\} \cup \{(r_4^u, t_1) \mid u \in U\} \\ &\subseteq \mathscr{R},\end{aligned} \tag{12}$$

where

$$\begin{aligned}&r_2^0(\varphi, \varphi \to \psi) = \psi, \quad t_2^*(\theta, \beta) = \theta \wedge \beta, \\ &r_2^*(\varphi \to \gamma, \varphi \to \psi) = \varphi \to (\gamma \wedge \psi), \\ &r_2^{\triangle}(\varphi \to \psi, \psi \to \gamma) = \varphi \to \gamma, \\ &r_1^{\theta_0}(\varphi) = \theta_0 \to \varphi, \quad t_1^{\theta_0}(\alpha) = \theta_0 \to \alpha, \\ &r_1^u(\varphi) = (\mathbf{Q}_u x)\varphi, \quad t_1(\theta) = \theta, \\ &r_2^u(\varphi \to \psi) = \varphi \to (\mathbf{Q}_u x)\psi, \ x \text{ is not free in } \varphi, \\ &r_3^u(\varphi \to \psi) = (\mathbf{Q}_u x)\varphi \to \psi, \ x \text{ is not free in } \psi, \\ &r_4^u(\mathbf{Q}_u x)(\varphi \otimes \psi) = (\mathbf{Q}_u x)\varphi \otimes \psi, \ x \text{ is not free in } \psi.\end{aligned}$$

In the following, we use the set of inference rules $\mathscr{R}^*$ and the set of interpretations $\mathcal{I}_i$ for uncertain reasoning, where

$$\mathcal{I}_i = \{T \mid T \in \mathscr{F}_L(\mathcal{F}_f), T \text{ is } \alpha\text{-}i \text{ type closed w.r.t. } \mathscr{R}^*\}, \ i = \mathrm{I}, \mathrm{II}.$$

The following theorem shows that such selected $\mathscr{R}^*$ and $\mathcal{I}_i$ can guarantee the soundness and completeness of lattice-valued logic according to Theorem 9.

Theorem 10 [2] *Given* $\mathscr{R}$ *and* α. *If*

$$\mathcal{I} = \{T \mid T \in \mathscr{F}_L(\mathcal{F}_f), T \textit{ is } \alpha\textit{-}i \textit{ type closed w.r.t. } \mathscr{R}\},$$

then $C^{\beta,\tilde{X}}_{(C^{\emptyset}_{\mathcal{I}}, \mathscr{R}(\alpha-i))} \in \mathcal{I}$, i=I, II.

As for the truth-value field L, it should be selected according to real requirements. In this paper, in order to provide some ideas for dealing with qualitative information which are widely used in real-life evaluation problems, we take the algebraic structure for modeling linguistic terms, linguistic truth-valued lattice implication algebra (L-LIA) [16], [17], as the truth-value field. L-LIA is constructed from the product of two finite Łukasiewicz chain. One is a Łukasiewicz chain with two elements which are meta truth values, "true" and "false", and the other chain is the set of some modifiers, also know as linguistic hedges [19] such as "very," "less," "possibly," etc. The number of modifiers is always odd [16], [3], such 3, 5 or 9. For more information about L-LIA, please refer to [16].

So, suppose that there are two finite Łukasiewicz chain, $L_2 = \{b_1,\ b_2\}$ and $L_n = \{a_1,\ a_2,\ \cdots,\ a_n\}$, where $n \in \mathbb{N}^+$, an odd natural number. The product LIA produced by them is denoted as $L_{2n} = L_n \times L_2$, and the lattice-valued first-order logic whose truth-value field is L_{2n} is denoted as $\mathscr{L}_{2nf}$.

Then, there are three parameters, α, β and τ, whose values remain to be determined. From the properties of L_{vfl}, $\alpha \le \bigwedge_{\theta\in L}(\theta \vee \theta')$ can generally guarantee that $\mathscr{R}^*$ is α-i type sound w.r.t. $\mathcal{I}_i$. So, in the following, we pay more attention to the selection of the values of parameters β and τ.

4 Parameter Selection

Because of the importance of $\bigwedge_{\theta\in L}(\theta \vee \theta')$ as a threshold for the soundness of inference rule, we firstly find its concrete value in L_{2n}.

Lemma 11 *In the product LIA L_{2n},*

$$\bigwedge_{\theta\in L_{2n}}(\theta \vee \theta') = \begin{cases} (a_{\frac{n+1}{2}}, b_2), & n \text{ is odd}, \\ (a_{\frac{n}{2}+1}, b_2), & n \text{ is even}. \end{cases}$$

Proof. In fact,

$$\begin{aligned}\bigwedge_{\theta\in L_{2n}}(\theta \vee \theta') &= \bigwedge_{(a_i,b_j)\in L_{2n}}((a_i,b_j) \vee (a_i',b_j')) \\ &= \bigwedge_{(a_i,b_j)\in L_{2n}}((a_i \vee a_i'),\ (b_j \vee b_j')) \\ &= (\bigwedge_{a_i\in L_n}(a_i \vee a_i'),\ \bigwedge_{b_j\in L_2}(b_j \vee b_j')) \\ &= \begin{cases} (a_{\frac{n+1}{2}}, b_2), & n \text{ is odd}, \\ (a_{\frac{n}{2}+1}, b_2), & n \text{ is even}. \end{cases}\end{aligned}$$

In the following, we determine the values of parameters β and τ by applying the uncertain reasoning process to some typical conditions.

Theorem 12 *For any $X \in \mathscr{F}_L(\mathcal{F}_f)$, if*

$$X(\varphi) = \begin{cases} \varphi, & \varphi \in L_{2n}, \\ O, & \text{otherwise}, \end{cases}$$

where $\varphi \in \mathcal{F}_f$. *Then for any* $\beta \in L_{2n}$, X *is* I-i *(i.e.,* $\tau = 0$*) type consistent w.r.t.* $(\alpha, \beta, \mathcal{I}_i)$, *where* $\alpha \leq \bigwedge_{\theta \in L_{2n}} (\theta \vee \theta')$, $i = \mathrm{I}, \mathrm{II}$.

Proof. If $\alpha \leq \bigwedge_{\theta \in L_{2n}} (\theta \vee \theta')$, then it follows from the properties of L_{vfl} that $\mathcal{I}_H \subseteq \mathcal{I}_i$. So there exists $T_0 \in \mathcal{I}_H$, such that

$$\begin{aligned} C_{\mathcal{I}_i}^{\beta \otimes X}(\psi) &= \bigwedge_{T \in \mathcal{I}_i} [(\bigwedge_{\varphi \in \mathcal{F}_f} (\beta \otimes X(\varphi) \to T(\varphi))) \to T(\psi)] \\ &= \bigwedge_{T \in \mathcal{I}_i} [(\bigwedge_{(a_i, b_j) \in L} (\beta \otimes (a_i, b_j) \to T((a_i, b_j)))) \to T(\psi)] \\ &\leq [\bigwedge_{(a_i, b_j) \in L} (\beta \otimes (a_i, b_j) \to T_0((a_i, b_j)))] \to T_0(\psi) = T_0(\psi). \end{aligned}$$

Therefore, $C_{\mathcal{I}_i}^{\beta \otimes X}(\psi) \otimes C_{\mathcal{I}_i}^{\beta \otimes X}(\psi') \leq T_0(\psi) \otimes T_0(\psi') = O = I'$.
Hence, X is I-i type consistent w.r.t. $(\alpha, \beta, \mathcal{I}_i)$, $i = \mathrm{I}, \mathrm{II}$.

Theorem 13 *For any* $X \in \mathscr{F}_L(\mathcal{F}_f)$, *if*

$$X(\varphi) = \begin{cases} \varphi, \varphi \in L_{2n}, \\ \xi, \quad \textit{otherwise}, \end{cases}$$

where $\varphi \in \mathcal{F}_f$, $\xi \in L_{2n}$. *Then we can select* $\beta = \xi'$, *such that* X *is* I-i *type consistent w.r.t.* $(\alpha, \beta, \mathcal{I}_i)$, *where* $\alpha \leq \bigwedge_{\theta \in L_{2n}} (\theta \vee \theta')$, $i = \mathrm{I}, \mathrm{II}$.

Proof. Because $\alpha \leq \bigwedge_{\theta \in L_{2n}} (\theta \vee \theta')$, then $\mathcal{I}_H \subseteq \mathcal{I}_i$. There exits $T_0 \in \mathcal{I}_H$, such that

$$\begin{aligned} C_{\mathcal{I}_i}^{\beta \otimes X}(\psi) &= \bigwedge_{T \in \mathcal{I}_i} [(\bigwedge_{\varphi \in \mathcal{F}_f} (\xi' \otimes X(\varphi) \to T(\varphi))) \to T(\psi)] \\ &= \bigwedge_{T \in \mathcal{I}_i} [(\bigwedge_{\mu \nleq \xi} (\xi' \otimes \mu \to T(\mu))) \to T(\psi)] \\ &\leq T_0(\psi). \end{aligned}$$

Therefore,

$$C_{\mathcal{I}_i}^{\beta \otimes X}(\psi) \otimes C_{\mathcal{I}_i}^{\beta \otimes X}(\psi') \leq T_0(\psi) \otimes T_0(\psi') = O = I'.$$

Hence, X is I-i type consistent w.r.t. $(\alpha, \beta, \mathcal{I}_i)$, $i = \mathrm{I}, \mathrm{II}$.

The following theorem can be obtained easily from Theorems 10 and 13.

Theorem 14 *If* $Y, \tilde{X} \subseteq X$ *in the uncertain reasoning model (10), and*

$$X(\varphi) = \begin{cases} \varphi, \varphi \in L_{2n}, \\ \xi, \ \textit{otherwise}, \end{cases}$$

where $\varphi \in \mathcal{F}_f$, $\xi \in L_{2n}$. *Then the above selected* $\mathscr{R}^*$ *and* $\mathcal{I}_i$ *makes the uncertain reasoning model (10)*$(\alpha, \xi', 0, \mathcal{I}_i)$*-i type regular, where* $\alpha \leq \bigwedge_{\theta \in L_{2n}}(\theta \vee \theta')$, $i = \mathrm{I}, \mathrm{II}$. *Therefore, we can get the uncertain reasoning consequence* $\widetilde{Y} = C_{\mathcal{I}_i}^{\beta \otimes \widetilde{X}}$, *which can also be obtained through a strict formal deduction in* $\mathscr{L}_{2nf}$.

If Y and $\tilde{X}$ take the same forms as X in the above theorem, then we can get the following theorem.

Theorem 15 *If in the uncertain reasoning model (10),* X, Y, $\tilde{X}$ *are given in the following forms:*

$$X(\varphi) = \begin{cases} \varphi, \ \varphi \in L_{2n}, \\ \xi_1, \ \textit{otherwise}; \end{cases}$$
$$Y(\varphi) = \begin{cases} \varphi, \ \varphi \in L_{2n}, \\ \xi_2, \ \textit{otherwise}; \end{cases}$$
$$\tilde{X}(\varphi) = \begin{cases} \varphi, \ \varphi \in L_{2n}, \\ \xi_3, \ \textit{otherwise}; \end{cases}$$

where $\varphi \in \mathcal{F}_f$, $\xi_1, \xi_3, \xi_3 \in L_{2n}$. *Let* $\beta = \xi_1' \wedge \xi_2' \wedge \xi_3'$, $\tau = \bigwedge\{\eta \in L_{2n} \mid \eta' \otimes \xi_2 \leq \xi_1\}$, *then the uncertain reasoning model (10) is* $(\alpha, \beta, \tau, \mathcal{I}_i)$*-i type regular, where* $\alpha \leq \bigwedge_{\theta \in L_{2n}}(\theta \vee \theta')$, $i = \mathrm{I}, \mathrm{II}$. *Then the uncertain reasoning consequence* $\widetilde{Y} = C_{\mathcal{I}_i}^{\beta \otimes \widetilde{X}}$, *which can also be obtained by a strict formal deduction in* $\mathscr{L}_{2nf}$.

Furthermore, we can get the following theorem if X, Y, $\tilde{X}$ take more general forms.

Theorem 16 *If in the uncertain reasoning model (10),* X, Y, $\tilde{X}$ *are given as:*

$$X(\varphi) = \begin{cases} c_1, \ \varphi = \varphi_1, \\ \vdots \\ c_m, \varphi = \varphi_m, \\ O, \ \textit{otherwise}; \end{cases}$$
$$Y(\psi) = \begin{cases} d_1, \ \psi = \psi_1, \\ \vdots \\ d_l, \ \psi = \psi_l, \\ O, \ \textit{otherwise}; \end{cases}$$
$$\tilde{X}(\gamma) = \begin{cases} e_1, \ \gamma = \gamma_1, \\ \vdots \\ e_s, \ \gamma = \gamma_s, \\ O, \ \textit{otherwise}, \end{cases}$$

where $m, l, s \in \mathbb{N}^+$, φ, φ_i, ψ, ψ_j, γ, $\gamma_k \in \mathcal{F}_f$, c_i, d_j, $e_k \in L_{2n}$, $i = 1, \cdots, m$, $j = 1, \cdots, l$, $k = 1, \cdots, s$. *Then we can choose* $\beta_X = c_1' \wedge \cdots \wedge c_m'$, $\beta_Y =$

$d_1' \wedge \cdots \wedge d_l'$, $\beta_{\tilde{X}} = e_1' \wedge \cdots \wedge e_s'$, and $\beta = \beta_X \wedge \beta_Y \wedge \beta_{\tilde{X}}$. If there exists $\tau \in L_{2n}$, such that $C_{\mathcal{I}}^X \supseteq \tau' \otimes Y$, then the uncertain reasoning model (10) is $(\alpha, \beta, \tau, \mathcal{I}_i)$-i type regular, where $\alpha \leq \bigwedge_{\theta \in L_{2n}} (\theta \vee \theta')$, $i = \mathrm{I}, \mathrm{II}$. Then the uncertain reasoning consequence $\widetilde{Y} = C_{\mathcal{I}_i}^{\beta \otimes \widetilde{X}}$, which can also be obtained by a strict formal deduction in $\mathscr{L}_{2nf}$.

5 An Illustrative Example

In this section, we will give a simple example of evaluation of cars to show how the proposed reasoning approach can be used in decision making with uncertainty.

Suppose that we are evaluating three kinds of cars: Benz (x_1), Toyota (x_2) and Ford (x_3), and there are four criteria or attributes: price (φ_1), safety (φ_2), comfort (φ_3) and fuel economy (φ_4). The truth-value field for modeling linguistic judgments is chosen as the L-LIA $L_{9\times 2}$ in [16] with nine modifiers: slightly (a_1), somewhat (a_2), rather (a_3), almost (a_4), exactly (a_5), quite (a_6), very (a_7), highly (a_8) and absolutely (a_9), and two prime terms: false (b_1) and true (b_2). The judgment of each criterion for each kind of car is given in Table 1, by taking a simple standardization of these natural expressed evaluations, e.g. the evaluation "the car is rather cheap" is transformed into "the price of the car is cheap" with a truth degree "rather true".

Table 1. Evaluation matrix of cars

	φ_1	φ_2	φ_3	φ_4
x_1	(a_6, b_1)	(a_7, b_2)	(a_7, b_2)	(a_3, b_1)
x_2	(a_3, b_2)	(a_2, b_1)	(a_3, b_2)	(a_7, b_2)
x_3	(a_2, b_2)	(a_2, b_2)	(a_2, b_2)	(a_2, b_2)

The evaluation values in Table 1 for car x_1, x_2, and x_3 are expressed as $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{X}_3$ respectively, *e.g.*, that for x_1 is

$$\widetilde{X}_1(\varphi) = \begin{cases} (a_6, b_1), \ \varphi = \varphi_1(x_1), \\ (a_7, b_2), \ \varphi = \varphi_2(x_1), \\ (a_7, b_2), \ \varphi = \varphi_3(x_1), \\ (a_3, b_1), \ \varphi = \varphi_4(x_1), \\ O, \qquad \text{otherwise}, \end{cases}$$

The decision rule is from our daily experience: "If the car is rather cheap, very safe, very comfortable and with quite good fuel economy, then the car is highly good", with a belief degree $\alpha = (a_5, b_2)$. Then the decision rule can be expressed as

$$\text{If } X \text{ then } Y,$$

where

$$X(\varphi) = \begin{cases} (a_7, b_2), \ \varphi = (\forall x)\varphi_1(x), \\ (a_7, b_2), \ \varphi = (\forall x)\varphi_2(x), \\ (a_7, b_2), \ \varphi = (\forall x)\varphi_3(x), \\ (a_7, b_2), \ \varphi = (\forall x)\varphi_4(x), \\ O, \qquad \text{otherwise}, \end{cases}$$

$$Y(\psi) = \begin{cases} (a_8, b_2), \ \psi = (\forall x)\psi_1(x), \\ O, \qquad \text{otherwise}. \end{cases}$$

The consistency levels of X, Y, and $\widetilde{X}_i$ $(i = 1, 2, 3)$ are all $\tau' = (a_7, b_2)$, and the belief degree of the reasoning process is chosen to be $\beta = (a_7, b_2)$. By applying Theorem 16, we can get the overall evaluation result for car x_1,

$$\widetilde{Y}_1(\psi(x_1)) = C_{\mathcal{I}_i}^{\beta \otimes \widetilde{X}} = (a_3, b_1).$$

Similarly, we can get the overall evaluation results $\widetilde{Y}_2(\psi(x_2)) = (a_3, b_2)$, $\widetilde{Y}_3(\psi(x_3)) = (a_2, b_2)$, for x_2 and x_3.

These results can be retransformed into natural language: car x_1 is rather bad, car x_2 is rather good, car x_3 is somewhat good, according to the provided criterions and observations. It can be seen by a simple comparison that x_2, *i.e.*, Toyota, may be a better choice among these three cars.

6 Conclusions

This paper proposed a parameterized uncertain reasoning approach with parameters for featuring uncertainty from different sources, which is a common phenomenon in many intelligent systems, based on a lattice-valued logic $\mathscr{L}_{2nf}$. We discussed some methods for selecting appropriate parameters during the uncertain reasoning process. Reasoning with different parameter values means reasoning with different degrees of belief and consistency. This proposed parameterized uncertain reasoning approach takes the advantage of direct reasoning with observed information to get the result, without the underlying numerical approximation needed by some other methods. An example for car evaluation was given to illustrate how the proposed uncertain reasoning approach work.

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References

1. Benferhat, S., Sossai, C.: Reasoning with Multiple-Source Information in a Possibilistic Logic Framework. Information Fusion. 7, 80-96 (2006)

2. Chen, S.W., Xu, Y.: Uncertainty Reasoning Based on Lattice-Valued First-Order Logic L_{vfl}. In: IEEE 2004 International Coference on Systems, Man, and Cybernetics, pp. 2237–2242. IEEE Press, New York (2004)
3. Chen, S.W., Xu, Y., Ma, J.: A Linguistic Truth-Valued Uncertainty Reasoning Model Based on Lattice-Valued Logic. In: Wang, L.P., Jin, Y.C. (eds.) FSKD 2005. LNAI, vol. 3613, pp. 276–283. Springer, Heidelberg (2005)
4. Cignoli, R., D'Ottaviano, I., Mundici, D.: Algebraic Foundations of Many-Valued Reasoning. Kluwer, Dordrecht (2000)
5. Gottwald, S.: A Treatise on Many-Valued Logics. Studies in Logic and Computation, vol. 9. Research Studies Press Ltd., Baldock (2001)
6. Hájek, P.: Metamathematics of Fuzzy Logic. Kluwer Academic Publishers, Dordrecht (1998)
7. Larsen, H.L., Yager, R.R.: A Framework for Fuzzy Recognition Technology. IEEE Trans. Syst. Man Cybern. Part C. 30, 65–76 (2000)
8. Novak, V., Perfilieva, I., Mojckojr, J.: Mathematical Principles of Fuzzy Logic. Kluwer Academic Publishers, Dordrecht (1999)
9. Pavelka, J.: On Fuzzy Logic I: Many-Valued Rules of Inference, II: Enriched Residuated Lattices and Semantics of Propositional Calculi, III: Semantical Completeness of Some Many-Valued Propositional Calculi. Zeitschr. F. Math. Logik und Grundlagend. Math. 25, 45–52, 119–134, 447–464 (1979)
10. Prado, R.P., Garcia-Galan, S., Munoz Exposito, J.E., Yuste, A.J.: Knowledge Acquisition in Fuzzy-Rule-Based Systems with Particle-Swarm Optimization. IEEE Trans. Fuzzy Syst. 18, 1083–1097 (2010)
11. Priest, G.: An Introduction to Non-Classical Logic (2nd Edition). Cambridge University Press, Cambridge (2008)
12. Sottara, D., Mello, P., Proctor, M.: A Configurable Rete-OO Engine for Reasoning with Different Types of Imperfect Information. IEEE Trans. Knowledge and Data Engineering. 22, 1535–1548 (2010)
13. Turunen, E.: Algebraic Structures in Fuzzy Logic. Fuzzy Sets Syst. 52, 181–188 (1992)
14. Xu, Y., Ruan, D., Liu, J.: Approximate Reasoning Based on Lattice-Valued Propositional Logic L_{vpl}. In: Ruan, D., Kerre, E.E. (eds.): Fuzzy Sets Theory and Applications. pp. 81–105. Kluwer Academic Publishers, Dordrecht (2000)
15. Xu, Y., Ruan, D., Qin, K.Y., Liu, J.: Lattice-Valued Logic: An Alternative Approach to Treat Fuzziness and Incomparability. Springer-Verlag, Berlin (2003)
16. Xu, Y., Chen, S.W., Ma, J.: Linguistic Truth-Valued Lattice Implication Algebra and Its Properties. In: IMACS 2006 Multiconference on "Computational Engineering in Systems Applications", pp. 1413–1418. IEEE Press, New York (2006)
17. Xu, Y., Chen, S.W., Liu, J., Ruan, D.: Weak Completeness of Resolution in a Linguistic Truth-Valued Propositional Logic. In: Castillo, O., Melin, P., Ross, O.M., Cruz, R.S., Pedrycz, W., Kacprzyk, J. (eds.) IFSA 2007. Advances in Soft Computin, vol. 42, pp. 358–366. Springer, Heidelberg (2007)
18. Xu, Y., Liu, J., Ruan, D., Li, X.B.: Determination of α-Resolution in Lattice-Valued First-Order Logic LF(X). Inform. Sci. doi:10.1016/j.ins.2010.03.024 (2010)
19. Zadeh, L.A.: The Concepts of a Linguistic Variable and Its Applications to Approximate Reasoning, Part I, II, III. Inform. Sci. 8, 199–249, 301–357, 9, 43–80 (1975)
20. Zhou, H.J., Wang, G.J., Zhou, W.: Consistency Degrees of Theories and Methods of Graded Reasoning in n-Valued R_0-Logic (NM-logic). Int. J. Approximate Reasoning. 43, 117-132 (2006)