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On the axiomatisability of priority

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In Memory of Sauro Tulipani

This paper studies the equational theory of bisimulation equivalence over the process algebra BCCSP extended with the priority operator of Baeten, Bergstra and Klop. We prove that, in the presence of an infinite set of actions, bisimulation equivalence has no finite, sound, ground-complete equational axiomatisation over that language. This negative result applies even if the syntax is extended with an arbitrary collection of auxiliary operators, and motivates the study of axiomatisations using equations with action predicates as conditions. In the presence of an infinite set of actions, it is shown that, in general, bisimulation equivalence has no finite, sound, ground-complete axiomatisation consisting of equations with action predicates as conditions over the language studied in this paper. Finally, sufficient conditions on the priority structure over actions are identified that lead to a finite, ground-complete axiomatisation of bisimulation equivalence using equations with action predicates as conditions.

1. Introduction

Programming and specification languages often include constructs to describe mode switches (see, for example, Mauw (1991) and Milner *et al.* (1987)). Indeed, some form of mode transfer in computation appears in operating systems in the guise of interrupts, in programming languages as exceptions, and in the behaviour of control programs and embedded systems as discrete 'mode switches' triggered by changes in the state of their environment. Such mode changes are often used to encode different levels of urgency amongst the actions that can be performed by a system as it computes, and implement variations on the notion of pre-emption.

In light of the ubiquitous nature of mode changes in computation, it is not surprising that classic process description languages include primitive operators to describe mode

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changes – for example, LOTOS (Brinksma 1985; ISO 1987) offers the so-called disruption operator – or have been extended with variations on mode transfer operators. Examples of such operators that may be added to the process algebra CCS are discussed by Milner in Milner (1989, pages 192–193), and Dsouza and Bloom offer in Dsouza and Bloom (1995) some discussion on the benefits of adding one of those, *viz*. the checkpointing operator, to CCS.

One of the most widely studied, and natural, notions used to implement different levels of urgency between system actions is priority. (A thorough and clear discussion of the different approaches to the study of priority in process description languages may be found in Cleaveland *et al.* (2001).) In this paper, we consider the well-known priority operator Θ studied by Baeten, Bergstra and Klop (Baeten *et al.* 1986) in the context of process algebra. (See Camilleri and Winskel (1985), Cleaveland and Hennessy (1990), Cleaveland *et al.* (2001) and Cleaveland *et al.* (1996) for later accounts of this operator in the setting of process description languages.) The priority operator Θ gives certain actions priority over others based on an irreflexive partial ordering relation $<$ over the set of actions. Intuitively, $a < b$ is interpreted as 'b has priority over a'. This means that, in the context of the priority operator Θ, action a is pre-empted by action b. For example, if p is some process that can initially perform both a and b, then $\Theta(p)$ will initially only be able to execute the action b.

In their classic paper Baeten *et al.* (1986), Baeten, Bergstra and Klop provided a sound and ground-complete axiomatisation for this operator modulo bisimulation equivalence. Their axiomatisation uses predicates on actions (to express priorities between actions) and one extra auxiliary operator. Bergstra showed in an earlier paper (Bergstra 1985) that, in case of a finite alphabet of actions, there exists a finite equational axiomatisation for Θ, without action predicates and help operators. So, if the set of actions is finite, neither equations with action predicates as conditions nor auxiliary operators, as used in Baeten *et al.* (1986), are actually necessary to obtain a finite axiomatisation of bisimulation equivalence over basic process description languages enriched with the priority operator. But, can Bergstra's positive result be extended to a setting with a countably infinite collection of actions? Or are equations with action predicates as conditions and auxiliary operators necessary to obtain a finite axiomatisation of bisimulation equivalence in the presence of an infinite collection of actions? (Note that infinite sets of actions are common in process calculi, and arise, for instance, in the setting of value- or name-passing calculi.) The aim of this paper is to provide a thorough answer to these questions in the setting of the process algebra BCCSP enriched with the priority operator Θ. In the case of an infinite alphabet, we permit the occurrence of action variables in axioms.

The process algebra BCCSP only contains basic process algebraic operators from CCS and CSP, but is sufficiently powerful to express all finite synchronisation trees. This paper considers the equational theory of BCCSP with the priority operator Θ from Baeten *et al.* (1986) modulo bisimulation equivalence. Our first main result is a theorem indicating that the use of equations with action predicates as conditions is indeed inevitable if we are to provide a finite axiomatisation of bisimulation equivalence over the basic process language we consider in this study. To this end, we prove that, in the case of an infinite alphabet and in the presence of at least one priority relation $a < b$

between a pair of actions, there is no finite equational axiomatisation for BCCSP enriched with the priority operator (Theorem 4.3). This result even applies if one is allowed to add an arbitrary collection of help operators to the syntax. Theorem 4.3 offers a very strong indication that the use of equations with action predicates as conditions is essential for axiomatising Θ, and cannot be circumvented by introducing auxiliary operators. (This is in contrast to the classic positive and negative results on the existence of finite equational axiomatisations for parallel composition offered in Bergstra and Klop (1984), Moller (1990a) and Moller (1990b).)

The idea underlying the proof of Theorem 4.3 is that for each finite sound equational axiomatisation E, there is a pair of actions c, d that does not occur in E. If c and d are incomparable, then

$$\Theta(c.\mathbf{0} + d.\mathbf{0}) \approx c.\mathbf{0} + d.\mathbf{0}$$

is sound modulo bisimulation equivalence. However, using a simple renaming argument, we show that a derivation of this equation from E would give rise to a derivation of the unsound equation $\Theta(a.\mathbf{0} + b.\mathbf{0}) \approx a.\mathbf{0} + b.\mathbf{0}$. Similarly, if $c < d$,

$$\Theta(c.\mathbf{0} + d.\mathbf{0}) \approx d.\mathbf{0}$$

is sound modulo bisimulation equivalence. But we prove that a derivation of this equation from E would give rise to a derivation of the unsound equation $\Theta(d.\mathbf{0} + c.\mathbf{0}) \approx c.\mathbf{0}$.

Having established that equations with action predicates as conditions are necessary in order to obtain a finite, ground-complete equational axiomatisation of bisimulation equivalence, we then proceed to investigate whether, in the presence of an infinite set of actions, this equivalence can be finitely axiomatised using equations with action predicates as conditions, but without auxiliary operators like the unless operator used in Baeten *et al.* (1986). We show that, in general, the answer to this question is negative. We do this by exhibiting a priority structure with respect to which bisimulation equivalence affords no finite, sound and ground-complete axiomatisation in terms of equations with action predicates as conditions (Theorem 5.6). This shows that, in general, the use of auxiliary operators is indeed necessary to axiomatise bisimulation equivalence finitely, even using equations with action predicates as conditions and over the simple language considered in this study. The priority structure used in the proof of Theorem 5.6 consists of actions a_i and b_i for $i \geqslant 1$ together with an action c, where $a_i < b_i < c$ for each $i \geqslant 1$. We prove that given a finite sound axiomatisation E consisting of equations with action predicates as conditions, the sound equation

$$\Theta(b_1.\mathbf{0} + \cdots + b_n.\mathbf{0}) \approx b_1.\mathbf{0} + \cdots + b_n.\mathbf{0}$$

cannot be derived from E, for a sufficiently large n.

In contrast with the aforementioned negative results, we exhibit a countably infinite, ground-complete axiomatisation for bisimulation equivalence over BCCSP with the priority operator in terms of equations with action predicates as conditions (Theorem 5.9). This axiomatisation suggests that, in general, infinite collections of pairwise incomparable actions with respect to the priority relation $<$ are the source of our negative result presented in Theorem 5.6. It is therefore natural to ask ourselves whether there are conditions that can be imposed on the poset of actions that are sufficient to guarantee that bisimulation equivalence be finitely axiomatisable using equations with action predicates as conditions, but without auxiliary operators. We conclude the technical developments in this paper by proposing some such sufficient conditions. The most general of these applies to all priority structures such that:

1 the collection of the sizes of the finite, maximal anti-chains is finite;
2 there are only finitely many infinite, maximal anti-chains; and
3 for each infinite, maximal anti-chain A, each element of A is above the same set of actions – that is, for each $a, b \in A$ and action c, we have that $c < a$ if and only if $c < b$.

Our results add the priority operator to the list of operators whose addition to a process algebra spoils finite axiomatisability modulo bisimulation equivalence; see, for example, Aceto *et al.* (2005), Aceto *et al.* (2006), Moller (1990a), Moller (1990b) and Sewell (1997) for other examples of non-finite axiomatisability results over process algebras. Notably, two mode transfer operators from Baeten and Bergstra (2000) are studied in Aceto *et al.* (2006) in the setting of the basic process algebra BPA. It is shown there that, even in the presence of just one action, the interrupt operator does not have a finite equational axiomatisation, while the disrupt operator does. In the interrupt operator, a process p can be interrupted by another process q; and upon termination of q, process p resumes its computation. In the disrupt operator, a process p can be pre-empted by another process q, after which the execution of p is aborted.

This paper is organised as follows. Section 2 contains some preliminaries. In Section 3, the finite axiomatisation for the priority operator Θ from Bergstra (1985) is presented. Section 4 contains the proof of a result to the effect that, in the case of an infinite alphabet, there is no finite equational axiomatisation for the priority operator modulo bisimulation equivalence, even in the presence of auxiliary operators. Finally, we show that, in the presence of an infinite set of actions, in general, bisimulation equivalence does not afford a finite axiomatisation in terms of equations with action predicates as conditions without the use of auxiliary operators (Section 5.1), and we identify sufficient conditions on the priority structure over actions that lead to the existence of a finite axiomatisation using equations with action predicates as conditions (Section 5.2).

2. Preliminaries

We begin by introducing the basic definitions and results on which the technical developments given later are based.

2.1. *The language* BCCSP_Θ

Act denotes a non-empty alphabet of atomic actions, with typical elements a, b, c, d, e. We assume an irreflexive, transitive partial ordering $<$ over *Act* to express priorities between actions. Intuitively, $a < b$ expresses the fact that the action b has priority over the action a. We say that actions $a_1, \ldots, a_n$ are *incomparable* if they are distinct and $a_i < a_j$ does not hold for all $1 \leqslant i, j \leqslant n$.

The language of processes we shall consider in this paper, which we will refer to as BCCSP_Θ from now on, is obtained by adding the unary priority operator Θ from Baeten *et al.* (1986) to the basic process algebra BCCSP (van Glabbeek 1990; van Glabbeek 2001). The language is given by the following grammar:

$$t ::= \mathbf{0} \mid a.t \mid t+t \mid \Theta(t) \mid x \mid \alpha.t\,,$$

where a ranges over Act, x is a process variable and α is an action variable. Process and action variables range over given, disjoint countably infinite sets. We use x, y, z to range over the collection of process variables, and α, β as typical action variables.

We use t, u, v to range over the collection of *open* process terms $\mathbb{T}(\mathrm{BCCSP}_\Theta)$. A process term is *closed* if it does not contain any variables, and p, q, r, range over the set of closed terms $\mathrm{T}(\mathrm{BCCSP}_\Theta)$. The *size* of a term is its length in function symbols.

Remark 2.1. The reader familiar with van Glabbeek (1990; 2001) might have already noticed that we consider a slightly extended syntax for BCCSP, in that we allow for the use of prefixing operators of the form $\alpha.\_$, where α is an action variable. The use of action variables is natural in the presence of infinite sets of actions, and will allow us to formulate stronger versions of the negative results to follow.

A substitution maps each process variable to a process term, and each action variable to an action or action variable. A substitution is *closed* if it maps process variables to closed process terms and action variables to actions. For every term t and substitution σ, the term obtained by replacing occurrences of process variables x and action variables α in t with $\sigma(x)$ and $\sigma(\alpha)$, respectively, is written $\sigma(t)$. Note that $\sigma(t)$ is closed if σ is closed. For example, $\sigma(\alpha.x) = a.\mathbf{0}$ if $\sigma(\alpha) = a$ and $\sigma(x) = \mathbf{0}$.

In general, for each signature Σ (that is, a collection of function symbols together with their arity), $\mathbb{T}(\Sigma)$ denotes the collection of open terms over Σ, and $\mathrm{T}(\Sigma)$ stands for the collection of closed terms over Σ. In Section 4, we shall consider signatures extending the signature for the language BCCSP_Θ.

The semantics of the operators is captured by the transition rules below, which give rise to Act-labelled transitions between closed terms. An *Act-labelled transition* between closed terms is a triple (p, a, p'), where p, p' are closed terms and $a \in Act$. From now on, and as usual, we shall use the suggestive notation $p \xrightarrow{a} p'$ instead of (p, a, p'). A *transition relation* is a collection of Act-labelled transitions.

The operational semantics for the language BCCSP_Θ is given by the labelled transition system

$$(\mathrm{T}(\mathrm{BCCSP}_\Theta), \to),$$

where the transition relation $\to$ is the unique supported model of the following rules in the sense of Bloom *et al.* (1995):

$$\frac{}{a.x \xrightarrow{a} x} \qquad \frac{x_1 \xrightarrow{a} y}{x_1 + x_2 \xrightarrow{a} y} \qquad \frac{x_2 \xrightarrow{a} y}{x_1 + x_2 \xrightarrow{a} y} \qquad \frac{x \xrightarrow{a} y \quad x \overset{b}{\nrightarrow} \ \text{ for } a < b}{\Theta(x) \xrightarrow{a} \Theta(y)}$$

where a ranges over Act. It is well known that the transition relation $\to$ is the one defined by structural induction over closed terms using the above rules.

Intuitively, closed terms in the language BCCSP_Θ represent finite process behaviours, where $\mathbf{0}$ does not exhibit any behaviour, $p+q$ is the non-deterministic choice between the behaviours of p and q, and $a.p$ executes action a to transform into p. Furthermore, the process graph of $\Theta(p)$ is obtained by eliminating all transitions $q \xrightarrow{a} q'$ from the process graph of p for which there is a transition $q \xrightarrow{b} q''$ with $a < b$.

We consider the language BCCSP_Θ modulo bisimulation equivalence.

Definition 2.2. A binary symmetric relation $\mathscr{R}$ over $\mathrm{T}(\mathrm{BCCSP}_\Theta)$ is a *bisimulation* if $p \mathrel{\mathscr{R}} q$ together with $p \xrightarrow{a} p'$ imply $q \xrightarrow{a} q'$ for some q' with $p' \mathrel{\mathscr{R}} q'$. We write $p \mathrel{\underline{\leftrightarrow}} q$ if there is a bisimulation relating p and q. The relation $\underline{\leftrightarrow}$ will be referred to as *bisimulation equivalence* or *bisimilarity*.

It is well known that $\underline{\leftrightarrow}$ is an equivalence relation. Moreover, the transition rules are in the GSOS format of Bloom *et al.* (1995). Hence, bisimulation equivalence is a congruence with respect to all the operators in the signature of BCCSP_Θ, meaning that $p \mathrel{\underline{\leftrightarrow}} q$ implies $C[p] \mathrel{\underline{\leftrightarrow}} C[q]$ for each BCCSP_Θ-context $C[\,]$.

We can therefore consider the algebra of the closed terms in $\mathrm{T}(\mathrm{BCCSP}_\Theta)$ modulo $\underline{\leftrightarrow}$. In Section 4, we shall give results that apply to any signature Σ that extends the signature of BCCSP_Θ. To this end, we tacitly assume that all of the new operators in Σ also preserve bisimulation equivalence, and are semantically interpreted as operations over finite synchronisation trees (Milner 1989).

2.2. *Equational logic*

An *axiom system* is a collection of equations $t \approx u$ over the language BCCSP_Θ. An equation $t \approx u$ is derivable from an axiom system E, notation $E \vdash t \approx u$, if it can be proved from the axioms in E using the rules of equational logic (*viz.* reflexivity, symmetry, transitivity, substitution and closure under BCCSP_Θ contexts).

$$\frac{}{t \approx t} \qquad \frac{t \approx u}{u \approx t} \qquad \frac{t \approx u \quad u \approx v}{t \approx v} \qquad \frac{t \approx u}{\sigma(t) \approx \sigma(u)}$$

$$\frac{t \approx u \quad t' \approx u'}{t + t' \approx u + u'} \qquad \frac{t \approx u}{a.t \approx a.u} \qquad \frac{t \approx u}{\alpha.t \approx \alpha.u} \qquad \frac{t \approx u}{\Theta(t) \approx \Theta(u)}$$

Without loss of generality, one may assume that substitutions happen first in equational proofs, that is, that the rule

$$\frac{t \approx u}{\sigma(t) \approx \sigma(u)}$$

may only be used when $t \approx u \in E$. Moreover, by postulating that for each axiom in E its symmetric counterpart is also present in E, we can disregard applications of symmetry in equational proofs. In the rest of this paper, we shall tacitly assume that our equational axiom systems are closed with respect to symmetry. Furthermore, it is well known (*cf.*, for example, Groote (1990, Section 2)) that if an equation relating two closed terms can be proved from an axiom system E, then there is a closed proof for it. (A proof is *closed* if it only mentions closed terms.) We shall only consider questions related to the provability

Table 1. *Axiomatisation for* $|Act| < \infty$

$x + y \approx y + x$	(A1)
$x + (y + z) \approx (x + y) + z$	(A2)
$x + x \approx x$	(A3)
$x + \mathbf{0} \approx x$	(A4)
$\Theta(\mathbf{0}) \approx \mathbf{0}$	(PR1)
$\Theta(a.x + a.y + z) \approx \Theta(a.x + z) + \Theta(a.y + z)$	(PR2)
$\Theta(a.x + b.y + z) \approx \Theta(b.y + z) \quad (a < b)$	(PR3)
$\Theta(a_1.x_1 + \cdots + a_n.x_n) \approx a_1.\Theta(x_1) + \cdots + a_n.\Theta(x_n) \quad (a_1, \ldots, a_n \text{ incomparable})$	(PR4)

of closed equations from an axiom system. Therefore, in light of the previous observation, we can restrict ourselves to considering closed proofs.

Definition 2.3. An equation $t \approx u$ is *sound* with respect to $\underline{\leftrightarrow}$ if $\sigma(t) \underline{\leftrightarrow} \sigma(u)$ holds for each closed substitution σ. An axiom system E is called *sound* over some language modulo $\underline{\leftrightarrow}$ if $E \vdash t \approx u$ implies $t \underline{\leftrightarrow} u$ for all terms t, u in the language. Conversely, E is called *ground-complete* if $p \underline{\leftrightarrow} q$ implies $E \vdash p \approx q$ for all *closed* terms p, q in the language.

Our order of business in the rest of this paper will be to present a thorough study of the equational theory of the language BCCSP_Θ modulo bisimulation equivalence. We will begin our investigation by considering the case in which the set of actions Act is finite in the following section. We then move on to investigate the equational properties of bisimulation equivalence over BCCSP_Θ when the set of actions is infinite (Sections 4 and 5).

3. $|Act| < \infty$

In this section, we assume that the action set is finite. The axiom system in Table 1 was put forward by Jan Bergstra in Bergstra (1985). Note that, in the case of a finite action set, this axiom system is finite, since then the axiom schemas PR2–4 give rise to finitely many equations.

Theorem 3.1 (Bergstra 1985). The axiom system consisting of the equations (A1)–(A4) and (PR1)–(PR4) is sound and ground-complete for BCCSP_Θ modulo $\underline{\leftrightarrow}$.

Proof. (Sketch) Since $\underline{\leftrightarrow}$ is a congruence with respect to BCCSP_Θ, soundness can be checked for each axiom separately. This is an easy exercise.

Next observe that, using (PR1)–(PR4), one can remove all occurrences of Θ from closed terms. Then ground-completeness follows from the well-known ground-completeness of (A1)–(A4) for BCCSP modulo $\underline{\leftrightarrow}$ (see, for example, Hennessy and Milner (1985)). □

In the rest of this paper, process terms are considered modulo associativity and commutativity of $+$. In other words, we do not distinguish between $t + u$ and $u + t$, or between $(t + u) + v$ and $t + (u + v)$. We use a *summation* $\sum_{i=1}^{n} t_i$ to denote $t_1 + \cdots + t_n$, where the empty sum represents $\mathbf{0}$. Such a summation is said to be in *head normal form* if each term t_i is of the form $a_i.t_i'$ or $\alpha_i.t_i'$ for some action a_i or action variable α_i, and term t_i'.

It is easy to see that modulo the axioms (A1) and (A2), every term t in the language BCCSP_Θ has the form $\sum_{i \in I} t_i$, for some finite index set I, and terms t_i $(i \in I)$ that do not have the form $t' + t''$. The terms t_i $(i \in I)$ will be referred to as the *summands* of t. For example, the term $\Theta(a.\mathbf{0} + b.\mathbf{0})$ has only itself as summand.

Remark 3.2. Note that the axiom system in Table 1 is not strong enough to prove all of the sound equations over the language BCCSP_Θ modulo bisimulation equivalence. For instance, one can check that the equation

$$\Theta(\Theta(x) + y) \approx \Theta(x + y)$$

is sound modulo bisimulation equivalence irrespective of the cardinality of the set of actions Act and the ordering relation $<$. However, this equation cannot be proved from those in Table 1.

4. $|Act| = \infty$

In this section, we deal with the case in which the action set is infinite. Our main result is that bisimulation equivalence does *not* afford a finite equational axiomatisation over the language BCCSP_Θ provided Act contains at least two actions a, b with $a < b$. (Otherwise, the equation $\Theta(x) \approx x$ would be sound, and the priority operator could be eliminated from all terms.) This negative result even applies if BCCSP_Θ is extended with an arbitrary collection of operators (over finite synchronisation trees) for which bisimulation equivalence is a congruence.

The idea behind the proof of our main result of this section is that a finite axiom system E can mention only finitely many action names. So, since Act is infinite, we can find a pair c, d of distinct actions that do not occur in E. If c and d are incomparable, the equation $\Theta(c.\mathbf{0} + d.\mathbf{0}) \approx c.\mathbf{0} + d.\mathbf{0}$ is sound; if $c < d$, then $\Theta(c.\mathbf{0} + d.\mathbf{0}) \approx d.\mathbf{0}$ is sound. In the first case, we show that an equational proof of $\Theta(c.\mathbf{0} + d.\mathbf{0}) \approx c.\mathbf{0} + d.\mathbf{0}$ from E would give rise to a proof of the unsound equation $\Theta(a.\mathbf{0} + b.\mathbf{0}) \approx a.\mathbf{0} + b.\mathbf{0}$ from E. This follows by a simple renaming argument, using the fact that c and d do not occur in E. Similarly, in the second case, a proof of $\Theta(c.\mathbf{0} + d.\mathbf{0}) \approx d.\mathbf{0}$ from E would give rise to a proof of the unsound equation $\Theta(d.\mathbf{0} + c.\mathbf{0}) \approx c.\mathbf{0}$ from E.

To present the formal proof of the aforementioned negative result, we first introduce the action renaming mentioned in the proof idea sketched above.

Definition 4.1. Let $A \subseteq Act$, and let Σ be a signature that includes the set of operators in BCCSP_Θ. We extend each renaming function $\rho : A \to Act$ to a function $\rho : \mathbb{T}(\Sigma) \to \mathbb{T}(\Sigma)$

as follows, where f is any operator that is not of the form $a.\_$:

$$
\begin{aligned}
\rho(\mathbf{0}) &\stackrel{\text{def}}{=} \mathbf{0} \\
\rho(a.t) &\stackrel{\text{def}}{=} \begin{cases} \rho(a).\rho(t) \text{ if } a \in A \\ a.\rho(t) \text{ if } a \notin A \end{cases} \\
\rho(f(t_1,\ldots,t_n)) &\stackrel{\text{def}}{=} f(\rho(t_1),\ldots,\rho(t_n)) \\
\rho(x) &\stackrel{\text{def}}{=} x \\
\rho(\alpha.t) &\stackrel{\text{def}}{=} \alpha.\rho(t)\,.
\end{aligned}
$$

For each substitution σ, the substitution $\rho(\sigma)$ is defined by $\rho(\sigma)(x) \stackrel{\text{def}}{=} \rho(\sigma(x))$ and

$$
\rho(\sigma)(\alpha) \stackrel{\text{def}}{=} \begin{cases} \rho(\sigma(\alpha)) & \text{if } \sigma(\alpha) \in A \\ \sigma(\alpha) & \text{otherwise.} \end{cases}
$$

The following lemma states that the renaming of actions that are not mentioned in an axiom system E preserves provability.

Lemma 4.2. Let $A \subseteq Act$ and $\rho : A \to Act$. Let Σ be a signature that includes the set of operators in BCCSP_Θ, and E be a collection of equations over Σ, and assume that all of the actions $a \in A$ do not occur in E. Then $E \vdash p \approx q$ implies $E \vdash \rho(p) \approx \rho(q)$.

Proof. The proof is by induction on the depth of a closed proof of the equation $p \approx q$ from E. We proceed by a case analysis on the last rule used in the proof of $p \approx q$ from E. The case of reflexivity is trivial, and that of transitivity follows immediately by using the induction hypothesis, so we will only consider the other cases, namely the instantiation of an axiom and closure under contexts (since we are dealing with closed proofs, closure with respect to prefixing by action variables need not be considered):

— Case $E \vdash p \approx q$ because $\sigma(t) = p$ and $\sigma(u) = q$ for some equation $t \approx u \in E$ and closed substitution σ. Then $\rho(p) = \rho(\sigma(t)) = \rho(\sigma)(\rho(t))$. According to the proviso of the lemma, no action $a \in A$ occurs in t, so it is clear that $\rho(t) = t$. Similarly, $\rho(q) = \rho(\sigma(u)) = \rho(\sigma)(\rho(u)) = \rho(\sigma)(u)$. Since $t \approx u \in E$, by substitution instance, $E \vdash \rho(\sigma)(t) \approx \rho(\sigma)(u)$. In other words, $E \vdash \rho(p) \approx \rho(q)$, which was to be shown.

— Case $E \vdash p \approx q$ because $p = a.p'$ and $q = a.q'$ where $E \vdash p' \approx q'$. If $a \in A$, then $\rho(p) = \rho(a).\rho(p')$ and $\rho(q) = \rho(a).\rho(q')$; otherwise, $\rho(p) = a.\rho(p')$ and $\rho(q) = a.\rho(q')$. In either case, by induction, $E \vdash \rho(p') \approx \rho(q')$. By context closure, $E \vdash \rho(p) \approx \rho(q)$.

— Case $E \vdash p \approx q$ because $p = f(p_1,\ldots,p_n)$ and $q = f(q_1,\ldots,q_n)$, for some operator f in the signature that is not of the form $a.\_$, where $E \vdash p_i \approx q_i$ for $i = 1,\ldots,n$. By definition, $\rho(p) = f(\rho(p_1),\ldots,\rho(p_n))$ and $\rho(q) = f(\rho(q_1),\ldots,\rho(q_n))$. By induction, $E \vdash \rho(p_i) \approx \rho(q_i)$ for $i = 1,\ldots,n$, and by context closure, $E \vdash \rho(p) \approx \rho(q)$. □

We are now in a position to show the first main result of this paper.

Theorem 4.3. Let $|Act| = \infty$, and $a < b$ for some $a, b \in Act$. Let Σ be a signature consisting of the operators in BCCSP_Θ, together with auxiliary operators for which bisimulation equivalence is a congruence. Then bisimulation equivalence has no finite, sound and ground-complete axiomatisation over $\mathrm{T}(\Sigma)$.

Proof. We need to show that no finite axiom system is both sound and ground-complete for $\mathrm{T}(\Sigma)$ modulo $\underline{\leftrightarrow}$. Let E be a finite axiom system over $\mathrm{T}(\Sigma)$ that is sound modulo $\underline{\leftrightarrow}$. Fix a pair of distinct actions $c, d \in Act$ that do not occur in E. We can select c, d such that either they are incomparable, or $c < d$. In the first case, the following equation is sound modulo $\underline{\leftrightarrow}$:

$$\Theta(c.\mathbf{0} + d.\mathbf{0}) \approx c.\mathbf{0} + d.\mathbf{0}\,.$$

Assume, in order to show a contradiction, that this equation can be derived from E. Consider the renaming function ρ defined as $\rho(c) = a$ and $\rho(d) = b$. Since neither c nor d occurs in E, Lemma 4.2 gives $E \vdash \rho(\Theta(c.\mathbf{0} + d.\mathbf{0})) \approx \rho(c.\mathbf{0} + d.\mathbf{0})$. That is, $E \vdash \Theta(a.\mathbf{0} + b.\mathbf{0}) \approx a.\mathbf{0} + b.\mathbf{0}$, which is not sound modulo $\underline{\leftrightarrow}$, since $a < b$. This contradicts the soundness of E.

In the second case, the following equation is sound modulo $\underline{\leftrightarrow}$:

$$\Theta(c.\mathbf{0} + d.\mathbf{0}) \approx d.\mathbf{0}.$$

Again, assume in order to show a contradiction that this equation can be derived from E. Consider the renaming function ρ defined as $\rho(c) = d$ and $\rho(d) = c$. Since neither c nor d occurs in E, Lemma 4.2 gives $E \vdash \rho(\Theta(c.\mathbf{0} + d.\mathbf{0})) \approx \rho(d.\mathbf{0})$. That is, $E \vdash \Theta(d.\mathbf{0} + c.\mathbf{0}) \approx c.\mathbf{0}$, which is not sound modulo $\underline{\leftrightarrow}$. Once more, this contradicts the soundness of E.

In either case, we can conclude that the axiom system E is not ground-complete. □

5. Axiomatising priority over an infinite action set, conditionally

Theorem 4.3 offers very strong evidence that, in the presence of an infinite set of actions, equational logic is inherently not sufficiently powerful to achieve a finite axiomatisation of bisimilarity over closed terms in the language BCCSP_Θ. Indeed, that result holds true even in the presence of an arbitrary number of auxiliary operators.

In the presence of action variables, it is natural to view our language as consisting of two sorts: one for actions and the other for processes. This is all the more true because the set of actions has the structure of a partial order, and we would like to express axioms over processes that reflect the influence that this poset structure on actions has on the behaviour of processes. When our set of actions is finite, this can be done by means of a finite number of equations that are instances of (PR3) and (PR4) in Table 1.

In the presence of an infinite action set, however, the axiom schemas (PR3) and (PR4), as well as (PR2), have infinitely many instances. One way to try and capture their effects finitely is to take seriously the idea that, in the presence of action variables, the equation schemas (PR3) and (PR4) can be phrased as *equations with action predicates as conditions* thus:

$$(\alpha < \beta) \Rightarrow \qquad \text{(CPR3)}$$
$$\Theta(\alpha.x + \beta.y + z) \;\approx\; \Theta(\beta.y + z)$$

$$\Big(\bigwedge_{1 \leqslant i,j \leqslant n} \neg(\alpha_i < \alpha_j)\Big) \Rightarrow \qquad \text{(CPR}4_n\text{)}$$
$$\Theta(\alpha_1.x_1 + \cdots + \alpha_n.x_n) \;\approx\; \alpha_1.\Theta(x_1) + \cdots + \alpha_n.\Theta(x_n) \quad (n \geqslant 0).$$

In both of the above equations, we use predicates over actions to restrict the applicability of the equation on the right-hand side of the implication. In general, in the rest of this paper we shall consider equations of the form

$$P \Rightarrow t \approx u,$$

where P is a predicate over actions, and $t \approx u$ is an equation over the language BCCSP$_\Theta$.

In the following, we shall take a semantic view of predicates over actions. An action predicate P will be identified simply with the collection of closed substitutions that satisfy it – with the proviso that two closed substitutions that agree over the collection of action variables are either both in P or neither of them is. As we did above for the equations (CPR3) and (CPR4$_n$), we shall often express predicates over actions using formulae in first-order logic with equality and the binary relation symbol $<$. The definition of the collection of closed substitutions that satisfy a predicate P expressed using such formulae is entirely standard, and we omit the details. For example, a closed substitution σ satisfies the predicate $\alpha < \beta$ if and only if $\sigma(\alpha) < \sigma(\beta)$ holds in the poset $(Act, <)$. We will sometimes write $\sigma(P) = true$ if the closed substitution σ satisfies the predicate P. We say that a predicate is *satisfiable* if some closed substitution satisfies it. If P is a tautology, we simply write $t \approx u$. For instance, a version of equation (PR2) with action variables will be written

$$\Theta(\alpha.x + \alpha.y + z) \;\approx\; \Theta(\alpha.x + z) + \Theta(\alpha.y + z). \tag{CPR2}$$

Note that equation (PR1) in Table 1 is just (CPR4$_0$). Moreover, since $<$ is irreflexive, equation (CPR4$_1$) reduces to

$$\Theta(\alpha.x) \;\approx\; \alpha.\Theta(x). \tag{1}$$

(Note that the above equation can be derived from each of the (CPR4$_n$) with $n \geqslant 1$ and axiom (A3) in Table 1.)

An equation of the form $P \Rightarrow t \approx u$ is *sound* with respect to bisimilarity, if $\sigma(t) \leftrightarrow \sigma(u)$ holds for each closed substitution σ that satisfies the predicate P. It is not hard to see that we have the following lemma.

Lemma 5.1. For each partial order of actions $(Act, <)$, the equations (CPR2), (CPR3) and (CPR4$_n$) with $(n \geqslant 0)$ are sound modulo bisimilarity over the language BCCSP$_\Theta$.

A proof in conditional equational logic of an equation from a set E of axioms with action predicates as conditions uses the same rules presented in Section 2.2. However, the rule for substitution instance now reads

$$\frac{P \Rightarrow t \approx u}{\sigma(t) \approx \sigma(u)} \quad (\sigma(P) = true),$$

where $P \Rightarrow t \approx u$ is one of the equations with action predicates as conditions in the set E. Again, by postulating that for each equation of the form $P \Rightarrow (t \approx u)$ in E its symmetric counterpart $P \Rightarrow (u \approx t)$ is also present in E, we can disregard applications of symmetry in conditional equational proofs.

A natural question to ask at this point, and one that we will address in the rest of this paper, is whether, unlike standard equational logic, equations with action predicates

Table 2. *Axioms for Θ in the presence of $\lhd$*

$$
\begin{array}{rcl}
\Theta(\alpha.x) & \approx & \alpha.x \\
\Theta(\mathbf{0}) & \approx & \mathbf{0} \\
\Theta(x+y) & \approx & (\Theta(x) \lhd y) + (\Theta(y) \lhd x) \\
\neg(\alpha < \beta) \;\Rightarrow\; (\alpha.x) \lhd (\beta.y) & \approx & \alpha.x \\
(\alpha < \beta) \;\Rightarrow\; (\alpha.x) \lhd (\beta.y) & \approx & \mathbf{0} \\
(\alpha.x) \lhd \mathbf{0} & \approx & \alpha.x \\
\mathbf{0} \lhd (\alpha.x) & \approx & \mathbf{0} \\
(x+y) \lhd z & \approx & (x \lhd z) + (y \lhd z) \\
x \lhd (y+z) & \approx & (x \lhd y) \lhd z
\end{array}
$$

as conditions suffice to obtain a finite, ground-complete axiomatisation of bisimulation equivalence over the language BCCSP_Θ.

In their classic paper (Baeten *et al.* 1986), Baeten, Bergstra and Klop offered a finite, ground-complete axiomatisation of bisimilarity over the language BPA_δ with the priority operator that employs equations with action predicates as conditions. Their axiomatisation, however, relied upon the introduction of a binary auxiliary operator, the so-called *unless* operator $\lhd$. Operationally, the behaviour of the unless operator is specified by the rules

$$\frac{x \xrightarrow{a} x' \quad y \not\xrightarrow{b} \text{ for } a < b}{x \lhd y \xrightarrow{a} x'},$$

where $a \in Act$.

In the setting of BCCSP_Θ, and using action variables instead of concrete action names, the relation between the priority operator and the unless operator is expressed by the axioms in Table 2. It is not too hard to see that those axioms, together with (A1)–(A4) in Table 1, yield a ground-complete, finite axiomatisation of bisimulation equivalence. Therefore, even in the presence of an infinite set of actions, bisimulation equivalence affords a finite, ground-complete axiomatisation using equations with action predicates as conditions at the price of introducing a single auxiliary operator. But, we can ask: if the set of actions is infinite, is the use of an auxiliary operator like the unless operator really necessary to obtain a finite axiomatisability result for bisimulation equivalence over BCCSP_Θ using equations with action predicates as conditions? We will address this question in the following. In particular, we first show that, in general, the use of auxiliary operators is indeed necessary to obtain a finite, ground-complete axiomatisation of bisimulation equivalence using equations with action predicates as conditions. We do this in Section 5.1 by exhibiting a poset of actions for which no finite set of sound equations with action predicates as conditions is ground-complete with respect to bisimulation equivalence over BCCSP_Θ. This negative result, however, does not entail that, in the presence of an infinite set of actions, auxiliary operators are always needed to give a finite, ground-complete axiomatisation of bisimulation equivalence over the language BCCSP_Θ. In fact, we then isolate sufficient conditions on the priority structure over actions that guarantee the finite axiomatisability of bisimulation equivalence over the language BCCSP_Θ using equations with action predicates as conditions (Section 5.2).

5.1. *A negative result*

Our order of business will now be to prove that, in the presence of an infinite set of actions, in general, auxiliary operators are indeed necessary if we are to obtain a finite ground-complete axiomatisation of bisimulation equivalence over the language BCCSP_Θ, even if we permit the use of equations of the form $P \;\Rightarrow\; (t \approx u)$. In this section, $Act = \{a_i, b_i \mid i \geqslant 1\} \cup \{c\}$, where $a_i < b_i < c$ for each $i \geqslant 1$, and these are the only inequalities. Moreover, for convenience, we consider terms not only modulo associativity and commutativity of $+$, but also modulo the sound equations $x + \mathbf{0} \approx x$ and $\Theta(\Theta(x) + y) \approx \Theta(x + y)$ – see Remark 3.2. So we can assume, without loss of generality, that terms contain neither redundant $\mathbf{0}$ summands nor nested occurrences of Θ.

We will prove the following claim, which will be used to argue that bisimulation equivalence has no finite, ground-complete axiomatisation consisting of equations with action predicates as conditions over the language BCCSP_Θ (Theorem 5.6).

Claim 5.2. Let E be a finite collection of equations with action predicates as conditions that is sound modulo $\underline{\leftrightarrow}$. Let $n \geqslant 2$ be larger than the size of any term in the equations of E. Then we cannot derive the following equation from E:

$$\Theta(\Phi_n) \approx \Phi_n,$$

where Φ_n denotes $\sum_{i=1}^{n} b_i.\mathbf{0}$.

Note that the equation above is sound modulo $\underline{\leftrightarrow}$ because the actions b_i $(i \geqslant 1)$ are pairwise incomparable.

First we establish a technical lemma.

Lemma 5.3. Let $P \Rightarrow t \approx u$ be an equation that is sound modulo $\underline{\leftrightarrow}$, where P is satisfiable. If some process variable x occurs as a summand in t, then x also occurs as a summand in u.

Proof. Since P is satisfiable, there exists a closed substitution σ such that $\sigma(P) = \mathit{true}$. Take some action $d \in Act$ that does not occur in $\sigma(u)$; such an action exists because Act is infinite. Consider the closed substitution σ' that maps x to $d.(b_1.\mathbf{0} + c.\mathbf{0})$, and all other process variables to $\mathbf{0}$, and that agrees with σ on action variables. As $P \Rightarrow t \approx u$ is sound modulo $\underline{\leftrightarrow}$ and $\sigma'(P) = \sigma(P) = \mathit{true}$, we have that $\sigma'(t) \,\underline{\leftrightarrow}\, \sigma'(u)$. Since x is a summand of t and $\sigma'(t) \xrightarrow{d} b_1.\mathbf{0} + c.\mathbf{0}$, it follows that $\sigma'(u) \xrightarrow{d} q \,\underline{\leftrightarrow}\, b_1.\mathbf{0} + c.\mathbf{0}$ for some q. Since d does not occur in $\sigma(u)$ and $b_1 < c$, it is not hard to see that x must be a summand of u. □

The following lemma forms the crux of the proof of our claim. It states a property of closed terms that holds for all of the closed instantiations of axioms in any finite, sound collection of equations with action predicates as conditions. As we shall see later, this property is also preserved by arbitrary conditional equational proofs from a finite, sound collection of equations with action predicates as conditions (Proposition 5.5).

Lemma 5.4. Let $P \Rightarrow t \approx u$ be sound modulo $\underline{\leftrightarrow}$. Let σ be a closed substitution with $\sigma(P) = true$. Assume that:

— n is larger than the size of t, where $n \geqslant 2$; and
— the summands of $\sigma(t)$ are all bisimilar to either Φ_n or $\mathbf{0}$.

Then the summands of $\sigma(u)$ are all bisimilar to either Φ_n or $\mathbf{0}$.

Proof. First suppose that all summands of $\sigma(t)$ are bisimilar to $\mathbf{0}$. Then $\sigma(t) \underline{\leftrightarrow} \mathbf{0}$, so the soundness of $P \Rightarrow t \approx u$ together with $\sigma(P) = true$ yields $\sigma(u) \underline{\leftrightarrow} \mathbf{0}$. This means that all summands of $\sigma(u)$ are bisimilar to $\mathbf{0}$, and we are done.

So we can assume that some summand of $\sigma(t)$ is bisimilar to Φ_n. Then $\sigma(t) \underline{\leftrightarrow} \sigma(u) \underline{\leftrightarrow} \Phi_n$, by the proviso of the lemma and the soundness of $P \Rightarrow t \approx u$.

We know that we can write $t = \sum_{i \in I} t_i$ and $u = \sum_{j \in J} u_j$ for some non-empty, finite index sets I and J, where the terms t_i and u_j are of the form x, $a.v$, $\alpha.v$ or $\Theta(v)$. By the proviso of the lemma, for each $i \in I$, the summands of $\sigma(t_i)$ are all bisimilar to Φ_n or $\mathbf{0}$. Since $n \geqslant 2$, for each $i \in I$, the term t_i is not of the form $a.v$ or $\alpha.v$. Hence either it is a process variable x, or it is of the form

$$\Theta\left(\sum_{\ell \in L_i} d_{i\ell}.t'_{i\ell} + \sum_{m \in M_i} \alpha_m.t''_{im} + \sum_{k \in K_i} z_{ik}\right)$$

(modulo the equations $x + \mathbf{0} \approx x$ and $\Theta(\Theta(x) + y) \approx \Theta(x + y)$). Let $I' \subseteq I$ be the set of indices of summands of t that have the above form. Observe that $K_i \neq \emptyset$ for each $i \in I'$ such that $\sigma(t_i)$ is bisimilar to Φ_n (because n is larger than the size of t). Note, moreover, that summands t_i of t having the above form such that $\sigma(t_i) \underline{\leftrightarrow} \mathbf{0}$ must have $L_i = M_i = \emptyset$, and for such summands $\sigma(z_{ik}) \underline{\leftrightarrow} \mathbf{0}$ for each $k \in K_i$.

Let us assume, in order to show a contradiction, that there is an index $j \in J$ such that $\sigma(u_j)$ has a summand that is not bisimilar to either Φ_n or $\mathbf{0}$. We proceed by a case analysis on the form of u_j:

1 Case $u_j = x$. By assumption, $\sigma(x)$ has a summand that is not bisimilar to either Φ_n or $\mathbf{0}$. Since $P \Rightarrow t \approx u$ is sound modulo $\underline{\leftrightarrow}$ and P is satisfiable (because $\sigma(P) = true$ by the proviso of the lemma), by Lemma 5.3, t also has x as a summand. Consequently, $\sigma(t)$ has a summand that is not bisimilar to either Φ_n or $\mathbf{0}$, which contradicts one of the assumptions of the lemma.

2 Case $u_j = a.u'_j$ or $u_j = \alpha.u'_j$. Since $\sigma(u) \underline{\leftrightarrow} \Phi_n$, we have that $a = b_h$ or $\sigma(\alpha) = b_h$ for some $1 \leqslant h \leqslant n$. Define the substitution σ' by

$$\sigma'(y) = \begin{cases} c.\mathbf{0} & \text{if } y = z_{ik} \text{ for some } i \in I' \text{ and } k \in K_i \\ \mathbf{0} & \text{otherwise} \end{cases}$$

for process variables y, and let σ' agree with σ on action variables. Then $\sigma'(t) \overset{b_h}{\nrightarrow}$, because:

— $c > b_h$;

— $K_i \neq \emptyset$ for every $i \in I'$ with $\sigma(t_i) \underline{\leftrightarrow} \Phi_n$;

— $L_i = M_i = \varnothing$ for every $i \in I'$ with $\sigma(t_i) \underline{\leftrightarrow} \mathbf{0}$; and

— t does not contain summands of the form $b_h.v$ or $\alpha.v$.

On the other hand, as σ and σ' agree on action variables, $\sigma'(u_j) \xrightarrow{b_h} \sigma'(u'_j)$. It follows that $\sigma'(u) \xrightarrow{b_h} \sigma'(u'_j)$, so $\sigma'(t) \not\underline{\leftrightarrow} \sigma'(u)$. Since $\sigma'(P) = \sigma(P) = \mathit{true}$, this contradicts the soundness of $P \Rightarrow t \approx u$ modulo $\underline{\leftrightarrow}$.

3 Case $u_j = \Theta(u')$. Then u_j consists of a single summand, so, by assumption, we have that $\sigma(u_j) \not\underline{\leftrightarrow} \Phi_n$ and $\sigma(u_j) \not\underline{\leftrightarrow} \mathbf{0}$.
Since $\sigma(u) \underline{\leftrightarrow} \Phi_n$, and terms are considered modulo the equations $x + \mathbf{0} \approx x$ and $\Theta(\Theta(x) + y) \approx \Theta(x + y)$, we can take u' to be of the form

$$\sum_{\ell \in L} e_\ell.u'_\ell + \sum_{m \in M} \beta_m.u''_m + \sum_{k \in K} y_k$$

for some finite index sets L, M, K. We distinguish two cases:

(a) For each $i \in I'$ with $\sigma(t_i) \not\underline{\leftrightarrow} \mathbf{0}$ there is a $k_i \in K_i$ such that z_{ik_i} is not a summand of u'.
Define the substitution σ' by

$$\sigma'(y) = \begin{cases} c.\mathbf{0} & \text{if } y = z_{ik_i} \text{ for some } i \in I' \text{ with } \sigma(t_i) \not\underline{\leftrightarrow} \mathbf{0}\text{, or} \\ & \text{if } y \text{ is a summand of } t \text{ with } \sigma(y) \not\underline{\leftrightarrow} \mathbf{0} \\ \sigma(y) & \text{otherwise} \end{cases}$$

for process variables y, and let σ' agree with σ on action variables. It is not hard to see that $\sigma'(t) \overset{b_i}{\nrightarrow}$ for $i = 1, \ldots, n$ (because $c > b_i$ and t has no summand of the form $a.v$ or $\alpha.v$). On the other hand, since $\sigma(u_j) \not\underline{\leftrightarrow} \mathbf{0}$ and $\sigma(u) \underline{\leftrightarrow} \Phi_n$, there is an h with $1 \leqslant h \leqslant n$ such that $\sigma(u') \xrightarrow{b_h}$. Furthermore, $\sigma(u') \overset{c}{\nrightarrow}$. By assumption, z_{ik_i} is not a summand of u' for each $i \in I'$ with $\sigma(t_i) \not\underline{\leftrightarrow} \mathbf{0}$. Moreover, for any variable summand y of t with $\sigma(y) \not\underline{\leftrightarrow} \mathbf{0}$, y is not a summand of u', because, by assumption, $\sigma(y) \underline{\leftrightarrow} \Phi_n$ while $\sigma(u') \not\underline{\leftrightarrow} \Phi_n$. So $\sigma(u') \xrightarrow{b_h}$ and $\sigma(u') \overset{c}{\nrightarrow}$ imply $\sigma'(u') \xrightarrow{b_h}$ and $\sigma'(u') \overset{c}{\nrightarrow}$. It follows that $\sigma'(u_j) \xrightarrow{b_h}$, so $\sigma'(u) \xrightarrow{b_h}$. Hence $\sigma'(t) \not\underline{\leftrightarrow} \sigma'(u)$. Since $\sigma'(P) = \sigma(P) = \mathit{true}$, this contradicts the fact that $P \Rightarrow t \approx u$ is sound modulo $\underline{\leftrightarrow}$.

(b) $\{z_{i_0k} \mid k \in K_{i_0}\} \subseteq \{y_k \mid k \in K\}$, for some $i_0 \in I'$ with $\sigma(t_{i_0}) \not\underline{\leftrightarrow} \mathbf{0}$.
In this case, K is non-empty since, as previously observed, K_{i_0} is non-empty. By the proviso of the lemma, $\sigma(t_{i_0}) \underline{\leftrightarrow} \Phi_n$, so (since n is larger than the size of t_{i_0}) there is a $k_0 \in K_{i_0}$ with $\sigma(z_{i_0k_0}) \not\underline{\leftrightarrow} \mathbf{0}$. Furthermore, by the assumption for case 3 of the proof, $\sigma(u_j) \not\underline{\leftrightarrow} \mathbf{0}$ and $\sigma(u_j) \not\underline{\leftrightarrow} \Phi_n$. Therefore, there is an h with $1 \leqslant h \leqslant n$ such that $\sigma(\Theta(u')) \overset{b_h}{\nrightarrow}$. Define the substitution σ' as

$$\sigma'(y) = \begin{cases} a_h.\mathbf{0} & \text{if } y = z_{i_0k_0} \\ \sigma(y) & \text{otherwise} \end{cases}$$

for process variables y, and let σ' agree with σ on action variables. We argue that $\sigma'(t) \overset{a_h}{\nrightarrow}$. To this end, observe first that, since $\sigma(\Theta(u')) \overset{b_h}{\nrightarrow}$, we have $\sigma(\sum_{k \in K} y_k) \overset{b_h}{\nrightarrow}$, so $\sigma(z_{i_0k_0}) \overset{b_h}{\nrightarrow}$. We are now ready to show that no summand of $\sigma'(t)$ affords an

a_h-labelled transition. We consider three exhaustive possibilities:

(i) Let $i \in I'$ with $z_{i_0k_0} \notin \{z_{ik} \mid k \in K_i\}$. Then clearly $\sigma'(t_i) \stackrel{a_h}{\nrightarrow}$.

(ii) Let $i \in I'$ with $z_{i_0k_0} \in \{z_{ik} \mid k \in K_i\}$. Then $\sigma(t_i) \not\underline{\leftrightarrow} \mathbf{0}$ because $\sigma(z_{i_0k_0}) \not\underline{\leftrightarrow} \mathbf{0}$, so, by assumption, $\sigma(t_i) \underline{\leftrightarrow} \Phi_n$. This implies $\sigma(t_i) \stackrel{b_h}{\rightarrow}$, so, since $\sigma(z_{i_0k_0}) \stackrel{b_h}{\nrightarrow}$, it follows that $\sigma'(t_i) \stackrel{b_h}{\rightarrow}$. Since the outermost function symbol of t_i is Θ, we can conclude that $\sigma'(t_i) \stackrel{a_h}{\nrightarrow}$.

(iii) Finally, since $\sigma(z_{i_0k_0}) \not\underline{\leftrightarrow} \mathbf{0}$ and $\sigma(z_{i_0k_0}) \stackrel{b_h}{\nrightarrow}$, the proviso of the lemma yields the fact that $z_{i_0k_0}$ cannot be a summand of t.

Since t has no other types of summands, from the three cases above we can conclude that $\sigma'(t) \stackrel{a_h}{\nrightarrow}$. On the other hand, $\sigma'(\Theta(u')) \stackrel{a_h}{\rightarrow}$ because $\sigma(\Theta(u')) \stackrel{b_h}{\nrightarrow}$ and $z_{i_0k_0} \in \{y_k \mid k \in K\}$. Hence $\sigma'(u) \stackrel{a_h}{\rightarrow}$, so $\sigma'(t) \not\underline{\leftrightarrow} \sigma'(u)$. Since $\sigma'(P) = \sigma(P) = true$, this contradicts the fact that $P \Rightarrow t \approx u$ is sound modulo $\underline{\leftrightarrow}$.

Summarising, the assumption that, for some $j \in J$, the term $\sigma(u_j)$ has a summand that is not bisimilar to either Φ_n or $\mathbf{0}$ leads to a contradiction. This completes the proof. □

The following proposition states that the property of closed instantiations of sound equations with action predicates as conditions mentioned in the above lemma is preserved under equational derivations from a finite collection of sound equations. This is the key to the promised proof of our claim.

Proposition 5.5. Let E be a finite collection of equations with action predicates as conditions that is sound modulo $\underline{\leftrightarrow}$. Let $n \geqslant 2$ be larger than the size of any term in the equations of E. Assume, furthermore, that

— $E \vdash p \approx q$; and
— the summands of p are all bisimilar to Φ_n or $\mathbf{0}$.

Then the summands of q are all bisimilar to Φ_n or $\mathbf{0}$.

Proof. We use induction on the depth of the closed proof of the equation $p \approx q$ from E. We proceed by a case analysis on the last rule used in the proof of $p \approx q$ from E:

— $E \vdash p \approx q$ because $\sigma(t) = p$ and $\sigma(u) = q$ for some equation $P \Rightarrow t \approx u \in E$ and closed substitution σ with $\sigma(P) = true$. The claim follows immediately from Lemma 5.4.
— $E \vdash p \approx q$ because $p = p' + p''$ and $q = q' + q''$ for some p', q', p'', q'' such that $E \vdash p' \approx q'$ and $E \vdash p'' \approx q''$. Since the summands of p are all bisimilar to Φ_n or $\mathbf{0}$, the same holds for p' and p''. By induction, the summands of q' and q'' are all bisimilar to Φ_n or $\mathbf{0}$. The claim now follows because the summands of q are those of q' and q''.
— $E \vdash p \approx q$ because $p = a.p'$ and $q = a.q'$ for some p', q' such that $E \vdash p' \approx q'$. This case is vacuous, because $n \geqslant 2$ and $p \underline{\leftrightarrow} \Phi_n$.
— $E \vdash p \approx q$ because $p = \alpha.p'$ and $q = \alpha.q'$ for some p', q' such that $E \vdash p' \approx q'$. This case is vacuous, because p and q are closed.
— $E \vdash p \approx q$ because $p = \Theta(p')$ and $q = \Theta(q')$ for some p', q' such that $E \vdash p' \approx q'$. The claim is immediate, because both p and q consist of a single summand, and $p \underline{\leftrightarrow} q$ by the soundness of E. □

Theorem 5.6. Let $Act = \{a_i, b_i \mid i \geqslant 1\} \cup \{c\}$, where $a_i < b_i < c$ for each $i \geqslant 1$, and these are the only inequalities. Then bisimulation equivalence has no ground-complete axiomatisation over BCCSP_Θ consisting of a finite set of sound equations with action predicates as conditions.

Proof. Let E be a finite collection of equations with action predicates as conditions that is sound modulo $\underline{\leftrightarrow}$. Let $n \geqslant 2$ be larger than the size of any term in the equations of E. According to Proposition 5.5, from E we cannot derive $\Theta(\Phi_n) \approx \Phi_n$. This equation is sound modulo $\underline{\leftrightarrow}$, and therefore E is not ground-complete. □

5.2. *Positive results*

In the previous section, we gave an example of a priority structure $(Act, <)$ with respect to which it is impossible to give a finite, ground-complete axiomatisation of bisimulation equivalence over BCCSP_Θ in terms of equations with action predicates as conditions, without recourse to auxiliary operators. That result, however, does not imply that auxiliary operators are always necessary to achieve a finite basis of equations with action predicates as conditions for bisimulation equivalence. Our aim in this section is to substantiate this claim by providing some general conditions over the priority structure $(Act, <)$ that are sufficient to guarantee the existence of a finite, ground-complete axiomatisation of bisimulation equivalence over BCCSP_Θ that uses equations with action predicates as conditions.

Definition 5.7. An anti-chain in a poset $(Act, <)$ is a subset of Act consisting of pairwise incomparable actions. The *width* of a poset $(Act, <)$ is the least upper bound of the cardinalities of its anti-chains. A poset $(Act, <)$ has *finite width* if its width is finite.

Example 5.8. The poset of actions we considered in Section 5.1 has uncountably many infinite, maximal anti-chains. (Each such anti-chain can, in fact, be obtained by picking exactly one of a_i and b_i for each $i \geqslant 1$.) The width of that poset is therefore infinite.

We now consider a countably infinite, ground-complete axiomatisation of bisimulation equivalence over BCCSP_Θ using equations with action predicates as conditions. Such an axiomatisation reduces to a finite one if the poset of actions has finite width.

Theorem 5.9. Let $(Act, <)$ be an infinite poset of actions. The following statements hold:

1 The axiom system consisting of the equations (CPR2), (CPR3) and (CPR4_n) ($n \geqslant 0$), together with equations (A1)–(A4) in Table 1 is ground-complete for bisimilarity over the language BCCSP_Θ.

2 Assuming that the width of $(Act, <)$ is k, the axiom system consisting of the equations (CPR2), (CPR3), and (CPR4_k), together with equations (A1)–(A4) and (PR1) in Table 1, is ground-complete for bisimilarity over the language BCCSP_Θ. Therefore, bisimilarity has a finite, ground-complete axiomatisation using equations with action predicates as conditions if $(Act, <)$ has finite width.

Proof. We will only present a sketch of the proof for statement (2). (The proof for statement (1) follows similar lines.)

First, observe that it is enough to show that if the cardinality of each anti-chain in $(Act, <)$ is at most k, the equations (CPR2), (CPR3), (CPR4$_k$) and (PR1) can be used to remove all occurrences of Θ from closed terms. Indeed, if we can do so, then ground-completeness follows from the well-known ground-completeness of (A1)–(A4) for BCCSP modulo $\underline{\leftrightarrow}$ (see, for example, Hennessy and Milner (1985)).

To prove that all occurrences of Θ can be removed from closed terms, assume that we have a closed term p that does not contain occurrences of Θ. We show that $\Theta(p)$ can be proved equal to a term q that does not contain occurrences of Θ by induction on the size of p. To this end, note that, modulo associativity and commutativity of $+$, the term p can be written $\sum_{i=1}^{n} a_i.p_i$ for some $n \geqslant 0$, actions a_i and closed terms p_i that do not contain occurrences of Θ.

If $n = 0$, equation (PR1) gives us that $\Theta(\mathbf{0}) \approx \mathbf{0}$, and we are done. If $n = 1$, the claim follows using (1) and the induction hypothesis. (Recall that, since $k \geqslant 1$, equation (1) is derivable from (CPR4$_k$).) Consider now the case when $n \geqslant 2$. We proceed by examining the following three sub-cases:

— there are i, j such that $1 \leqslant i < j \leqslant n$ and $a_i = a_j$;
— there are i, j such that $1 \leqslant i, j \leqslant n$ and $a_i < a_j$; and
— the collection of actions $\{a_1, \dots, a_n\}$ is an anti-chain in the poset $(Act, <)$.

The first two sub-cases are handled using the induction hypothesis, and equations (CPR2) and (CPR3), respectively.

If the proviso for the third sub-case applies, we know that $n \leqslant k$. Using equation (A3) if $n < k$, we can therefore reason as follows:

$$\begin{aligned}
\Theta\left(\sum_{i=1}^{n} a_i.p_i\right) &\approx \Theta\left(\sum_{i=1}^{n} a_i.p_i + \underbrace{a_n.p_n + \cdots + a_n.p_n}_{(k-n)\text{ times}}\right) \\
&\approx \sum_{i=1}^{n} a_i.\Theta(p_i) \quad \text{(by (CPR4}_k\text{) and possibly (A3))} \\
&\approx \sum_{i=1}^{n} a_i.q_i \quad \text{(by the induction hypothesis)}
\end{aligned}$$

for some closed terms $q_1, \dots, q_n$ that do not contain occurrences of Θ.

Using this result, a simple argument by structural induction over closed terms shows that each closed term in the language BCCSP$_\Theta$ is provably equal to one that does not contain occurrences of the Θ operator, and we are done. □

Thus, bisimilarity affords a finite, ground-complete axiomatisation that uses equations with action predicates as conditions if the poset $(Act, <)$ has finite width. (Moreover, the equations with action predicates as conditions making up the axiom systems used in Theorem 5.9 only involve predicates over actions that can be expressed as conjunctions of, possibly negated, atomic formulae of the form $\alpha < \beta$.) A natural question to ask at this point is whether this result holds for more general priority structures. We now proceed to address this question in some detail.

Let us begin by observing that there are priority structures with infinite anti-chains that *do* allow for a finite equational axiomatisation of bisimilarity over the language BCCSP_Θ. Consider, by way of an example, the flat priority structure $(\{\perp, a_0, a_1, \ldots\}, <)$, where the only ordering relations are given by $\perp < a_i$ for each $i \geqslant 0$. Membership of the countably infinite anti-chain $\{a_0, a_1, \ldots\}$ can be characterised by the predicate

$$P(\alpha) = \forall \beta.\ \neg(\alpha < \beta).$$

We can therefore write the following equation that allows us to reduce the number of summands within the scope of a Θ operator:

$$P(\alpha) \wedge P(\beta) \Rightarrow \Theta(\alpha.x + \beta.y + z) \ \approx\ \Theta(\alpha.x + z) + \Theta(\beta.y + z). \tag{2}$$

It is not hard to see that the above equation is sound. (In fact, the soundness of this equation will follow from the more general result in Lemma 5.12.) Moreover, following the lines of the proof sketch for Theorem 5.9(2), one can argue that, together with (PR1), (CPR2), (CPR3) and (1), this equation can be used to remove all occurrences of Θ from closed terms. Hence, we have the following proposition.

Proposition 5.10. Consider the priority poset $(\{\perp, a_0, a_1, \ldots\}, <)$, where the only ordering relations are given by $\perp < a_i$ for each $i \geqslant 0$. Then the axiom system consisting of the equations (2), (CPR2), (CPR3) and (1), together with equations (A1)–(A4) and (PR1) in Table 1, is ground-complete for bisimilarity over the language BCCSP_Θ.

As another example, consider the priority structure

$$\mathscr{A} = (\{a_0, a_1, \ldots\} \cup \{b_0, b_1, c\}, <),$$

where the relation $<$ is the least transitive relation satisfying

$$\begin{array}{lcll} b_i & < & a_j & \text{for all } i \in \{0,1\}, j \geqslant 0 \text{ and} \\ a_j & < & c & \text{for each } j \geqslant 0. \end{array}$$

This poset has one non-trivial maximal finite anti-chain, namely $\{b_0, b_1\}$, and one maximal countably infinite anti-chain, namely

$$A \ = \ \{a_0, a_1, \ldots\}.$$

Membership of A is characterised by the predicate P_A defined by

$$P_A(\alpha) \ = \ \exists \beta_1, \beta_2.\ \beta_1 < \alpha < \beta_2.$$

One can check that the instance of equation (2) associated with this predicate is sound. (Once again, the soundness of this equation will follow from the more general result in Lemma 5.12.) Moreover, following the lines of the proof sketch for Theorem 5.9(2), one can argue that, together with (PR1), (CPR2), (CPR3) and ($\mathrm{CPR4}_2$) (to handle the finite anti-chain $\{b_0, b_1\}$), this equation can be used to remove all occurrences of Θ from closed terms. Thus, we have the following proposition.

Proposition 5.11. Consider the priority poset $\mathscr{A}$. Then the axiom system consisting of equation (2) for predicate P_A, (CPR2), (CPR3) and ($\mathrm{CPR4}_2$), together with equations

(A1)–(A4) and (PR1) in Table 1, is ground-complete for bisimilarity over the language $BCCSP_\Theta$.

In both of the examples we have just presented, equation (2) plays a key role in that it allows us to reduce the size of terms in 'head normal form' having summands of the form $a.p$ and $b.q$ with a, b contained in an infinite anti-chain within the scope of a Θ operator. The following lemma states a necessary and sufficient condition on the infinite anti-chain that guarantees that axiom (2) is sound modulo bisimilarity.

Lemma 5.12. Let A be an anti-chain in the poset $(Act, <)$ whose membership is described by predicate P_A. Then the equation (2) for predicate P_A is sound modulo bisimilarity if and only if each element of A is above the same set of actions – that is, for each $a, b \in A$ and $c \in Act$, we have that $c < a$ if and only if $c < b$.

Proof. We first prove the 'if implication'. To this end, assume that $a, b \in A$ and p, q, r are closed terms in the language $BCCSP_\Theta$. We claim that

$$\Theta(a.p + b.q + r) \underline{\leftrightarrow} \Theta(a.p + r) + \Theta(b.q + r).$$

To see that this claim does hold, it is enough to observe that the following statements hold for each closed term p':

1 $\Theta(a.p + b.q + r) \xrightarrow{a} p'$ if and only if $\Theta(a.p + r) + \Theta(b.q + r) \xrightarrow{a} p'$;
2 $\Theta(a.p + b.q + r) \xrightarrow{b} p'$ if and only if $\Theta(a.p + r) + \Theta(b.q + r) \xrightarrow{b} p'$; and
3 $\Theta(a.p + b.q + r) \xrightarrow{c} p'$ if and only if $\Theta(a.p + r) + \Theta(b.q + r) \xrightarrow{c} p'$, for each action c different from a, b.

We only give a proof here for the last of these statements. To this end, assume first that $\Theta(a.p + b.q + r) \xrightarrow{c} p'$ for some action c different from a, b and closed term p'. Since c is different from a, b, there is a closed term r' such that:

— $p' = \Theta(r')$;
— $r \xrightarrow{c} r'$;
— $r \stackrel{d}{\nrightarrow}$ for each action d such that $c < d$; and
— neither $c < a$ nor $c < b$ holds.

It is now a simple matter to see that, for instance, $\Theta(a.p + r) \xrightarrow{c} p'$. This gives us that $\Theta(a.p + r) + \Theta(b.q + r) \xrightarrow{c} p'$, which was to be shown.

Conversely, suppose that $\Theta(a.p + r) + \Theta(b.q + r) \xrightarrow{c} p'$ for some action c different from a, b and closed term p'. Without loss of generality, we may assume that this is because $\Theta(a.p + r) \xrightarrow{c} p'$. Since c is different from a, b, there is a closed term r' such that:

— $p' = \Theta(r')$;
— $r \xrightarrow{c} r'$;
— $r \stackrel{d}{\nrightarrow}$ for each action d such that $c < d$; and
— $c < a$ does not hold.

Observe now that $c < b$ does not hold either, because a and b are above the same actions by the proviso of the lemma. It follows that $\Theta(a.p + b.q + r) \xrightarrow{c} p'$, which was to be shown.

To establish the 'only if implication', assume that A contains two distinct incomparable actions a and b that are *not* above the same set of actions. Suppose, without loss of generality, that $c < a$, but $c < b$ does not hold, for some action c. Then

$$\Theta(a.\mathbf{0} + b.\mathbf{0} + c.\mathbf{0}) \underline{\leftrightarrow} a.\mathbf{0} + b.\mathbf{0} \not\underline{\leftrightarrow}\ a.\mathbf{0} + b.\mathbf{0} + c.\mathbf{0} \underline{\leftrightarrow} \Theta(a.\mathbf{0} + c.\mathbf{0}) + \Theta(b.\mathbf{0} + c.\mathbf{0}).$$

(The last equivalence holds true because b and c must be incomparable, as $c < a$ and a and b are incomparable.) Therefore equation (2) for predicate P_A is not sound modulo bisimilarity. □

Remark 5.13. Let A, B be different, maximal anti-chains in the poset $(Act, <)$. Assume that all elements of A are above the same set of actions (that is, for each $a, b \in A$ and $c \in Act$, we have that $c < a$ if and only if $c < b$), and that all elements of B are also. Then A and B are disjoint.

To see this, assume, in order to show a contradiction, that $a \in A \cap B$. Since A and B are maximal anti-chains, neither is a subset of the other. Therefore, since $A \neq B$, there are actions b, c such that $b \in A - B$ and $c \in B - A$. It follows that a, b, c are above the same set of actions in Act. However, $b \notin B$. Therefore, since B is maximal, there must be some action $d \in B$ with $b < d$ or $d < b$. If $b < d$, we have that $b < a$ because $a, d \in B$ and each element of B is above the same actions. This contradicts the assumption that A is an anti-chain. If $d < b$, then, reasoning as above, we can reach a contradiction to the assumption that B is an anti-chain. Therefore, A and B must be disjoint.

Suppose that p is a closed term in head normal form whose set of initial actions is included in an infinite anti-chain satisfying the constraint in the statement of Lemma 5.12. Then the sound equation (2) offers a way of 'simplifying' the term $\Theta(p)$. The use of this axiom is the key to the proof of the following generalisation of Theorem 5.9(2), and of Propositions 5.10 and 5.11.

Theorem 5.14. Let $(Act, <)$ be an infinite poset of actions. Assume that:

1. the collection of the sizes of the finite, maximal anti-chains in $(Act, <)$ is finite;
2. $(Act, <)$ has finitely many infinite, maximal anti-chains; and
3. for each infinite, maximal anti-chain A in $(Act, <)$, each element of A is above the same set of actions – that is, for each $a, b \in A$ and $c \in Act$, we have that $c < a$ if and only if $c < b$.

Let k be the size of the largest finite, maximal anti-chain in $(Act, <)$, or 1 if all maximal anti-chains are infinite. Then the axiom system consisting of one instance of the equation (2) for predicate P_A for each infinite anti-chain A in $(Act, <)$, (CPR2), (CPR3) and (CPR4$_k$), together with equations (A1)–(A4) and (PR1) in Table 1, is ground-complete for bisimilarity over the language BCCSP$_\Theta$.

Proof. The soundness of the axiom system is easily established, using Lemma 5.12 for the instances of axiom (2). The completeness of the axiom system can be shown along the lines of the proof of Theorem 5.9. The key step in the argument is again to prove that each term $\Theta(\sum_{i=1}^{n} a_i.p_i)$, where the p_i do not contain occurrences of Θ, can be proved equal to a term q that does not contain occurrences of Θ by induction on the size of $\sum_{i=1}^{n} a_i.p_i$. This we do by considering several sub-cases depending on the number n of summands in $\sum_{i=1}^{n} a_i.p_i$.

If $n = 0$, the claim follows using (PR1). If $n = 1$, it is enough to use (1) and the induction hypothesis. (Recall that (1) is derivable from (CPR4$_k$).) If $n \geqslant 2$, we distinguish the following sub-cases:

— there are i, j such that $1 \leqslant i < j \leqslant n$ and $a_i = a_j$;
— there are i, j such that $1 \leqslant i, j \leqslant n$ and $a_i < a_j$;
— the collection of actions $\{a_1, \ldots, a_n\}$ is an anti-chain in the poset $(Act, <)$.

The first two sub-cases are handled using the induction hypothesis, and the equations with action predicates as conditions (CPR2) and (CPR3), respectively.

The last sub-case is handled using (CPR4$_k$) as in the proof of Theorem 5.9 if the set of actions $\{a_1, \ldots, a_n\}$ is included in a finite maximal anti-chain. Assume now that $\{a_1, \ldots, a_n\}$ is only included in an infinite maximal anti-chain, say A. (In fact, Remark 5.13 ensures that such an anti-chain A is unique.) Using the instance of equation (2) for predicate P_A and induction, the claim follows.

The rest of the proof follows along the same lines as the proof of Theorem 5.9, so is omitted. □

Remark 5.15. The priority structure we employed in our proof of Theorem 5.6 satisfies neither condition 2 nor condition 3 in the proviso of the above theorem.

In light of the above result, bisimilarity has a finite, ground-complete axiomatisation using equations with action predicates as conditions over the language BCCSP$_\Theta$ if the poset of actions satisfies the proviso of the above theorem. The above theorem therefore generalises Propositions 5.10 and 5.11. A further example of a priority structure that satisfies the conditions stated in Theorem 5.14 is one having a finite collection of 'priority levels' each consisting of an infinite set of actions – consider, for instance, the poset

$$(\{a_{ij} \mid 1 \leqslant i \leqslant N, j \geqslant 1\}, <),$$

where N is a positive integer and $a_{ij} < a_{hk}$ holds if and only if $i < h$.

We have not yet attempted a complete classification of the priority structures for which bisimulation equivalence affords a finite axiomatisation in terms of equations with action predicates as conditions over the language BCCSP$_\Theta$. This is probably a hard problem, which we leave for future research.

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