On the Complete Axiomatization for Prefix Iteration modulo Observation Congruence ⋆

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Abstract. Prefix iteration is a variation on the original binary version of the Kleene star operation P^*Q, obtained by restricting the first argument to be an atomic action. Aceto and Ingólfsdóttir provided an axiom system for observation congruence over basic CCS with prefix iteration. However hitherto the only direct completeness proof given for such a system is very long and technical. In this paper, we provide a new proof for the completeness of the axiom system in [3], which is a considerable simplification comparing to the original proof. Thus the open problem to find a direct completeness proof is closed.
Key Words: Process Algebra, Prefix Iteration, Observation Congruence, Axiomatization, Completeness.

1 Introduction

Kleene [13] defined a binary operator -*- in the context of finite automata, called *Kleene star* or *iteration*. Intuitively, the expression p^*q yields a solution for the recursive equation $X = p.X + q$. In other words, p^*q can choose to execute either p, after which it evolves into p^*q again, or q, after which it terminates. An advantage of the Kleene star is that on the one hand it can express recursion, but that on the other hand one can capture this operator in equational laws. Hence, one does not need meta-principles, such as Milner's *Unique Fixpoint Induction Principle* [15]. Kleene formulated several equations for this operator, e.g. $x^*y = x(x^*y) + y$.

The research literature on process theory has witnessed a resurgence of the interest in the study of Kleene star-like operations (cf. e.g. the papers [6][8][3][2][1]). Some researchers have studied the possibility of giving finite equational axiomatization of the bisimulation-like equivalence [17][14] over simple process algebras that include variations on Kleene's star operation.

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First of all, Sewell [18] shows that there does not exist a complete finite equational axiomatization for BPA_δ [5] with the Kleene star modulo strong bisimulation. In the light of this result, one way to obtain an equational axiomatization is to restrict the range of the terms that might occur at the left-hand side of the binary Kleene. The paper [6] might be the starting point of the work following this line. In that reference, Fokkink proposes a *finite*, complete equational axiomatization of strong bisimulation equivalence for $\mathrm{BCCS}^{p*}(A)$, that is, the language obtained by extending the fragment of Milner's CCS [14] containing the basic operations needed to express finite synchronization trees with *prefix iteration* a^*x, where a ranges over the atomic actions. Aceto and Groote [1] generalize this result to *string* iteration w^*x, where w ranges over strings of atomic actions whose length is smaller than some positive natural number N. Aceto and Ingólfsdóttir study prefix iteration in the presence of the silent step τ, in Milner's observation congruence. They extend the axiomatization from [6] with two standard equations for the silent step, and with three new equations which describe the interplay between the silent step and prefix iteration. Moreover, the completeness of their equational axiomatization w.r.t. observation congruence is shown. By term rewriting techniques, in [7] Fokkink presents a considerably shorter completeness proof for prefix iteration together with the silent step in rooted branching bisimulation equivalence from van Glabbeek and Weijland [12] in the setting of BPA [5] with the deadlock δ and empty process ϵ. In an unpublished paper, van Glabbeek shows that the completeness result in observation congruence from [3] follows from the completeness result in rooted branching bisimulation. The combination of the results of Fokkink and van Glabbeek leads to a considerably shorter completeness proof for prefix iteration in observation congruence than the one presented in [3]. As a conclusion, Aceto, Fokkink, van Glabbeek and Ingólfsdóttir have merged their three papers into one paper [2], which deals at once with weak, branching, delay, and η-bisimulation. Among other things, this paper presents a self-contained completeness proof for prefix iteration modulo rooted branching bisimulation.

However, to our knowledge, all the efforts to give a *direct* proof of the completeness theorem for prefix iteration in Milner's *observation congruence* [14] which is simpler than the one presented in [3] have failed. Let us quote what the researchers who are active in this area said:

- In [7], Fokkink wrote: "... This paper results from an attempt to try and shorten the long and technical completeness proof in [3]. Although this attempt was unsuccessful for observation congruence, it did yield a a considerably shorter completeness proof for prefix iteration together with the silent step in rooted branching bisimulation equivalence from van Glabbeek and Weijland [12]...".
- In [2], Aceto, Fokkink, van Glabbeek and Ingólfsdóttir wrote: "... All the authors' attempts to obtain a direct proof of the completeness theorem for weak congruence which is simpler than the one presented in [3] have been to no avail...". Note that here "weak congruence" refers to observation congruence in this paper.

This paper aims at giving a contribution to the study of complete equational axiomatization for Kleene star-like operations (concretely speaking, prefix iteration) from the point of view of process theory. We are motivated to shorten the long and technical completeness proof in [3], that is, to close the open problem in the sense of obtaining a direct proof of the completeness theorem for observation congruence which is simpler than Aceto and Ingólfsdóttir's. Following [3], we work on the language $\mathrm{BCCS}^{p*}(A_\tau)$. The axiom system for observation congruence has appeared in that reference, therefore our contribution lies in a much simpler and shorter proof. The main techniques used in this paper are standard. The source of simplicity, in our opinion, results from the extensive application of the so-called *Absorption Lemma* and the avoidance of well-known Hennessy Lemma [14]. Below we will examine this in more detail. It is worth pointing out that the following observation is not completely new, since it has been made by Fu and Yang in [10] for a language of mobile processes, π-calculus [16]. In particular, the promotion lemma is inspired by [10].

In the standard proof of the completeness theorem for observation congruence on finite CCS processes [14], one verifies first that every normal form process is provably equivalent to a saturated normal form process using the three τ-laws. Recall that a process P is saturated if, for each action α, one has that $P \xrightarrow{\alpha} P'$ whenever $P \overset{\alpha}{\Rightarrow} P'$. It follows that P is in saturated normal form if and only if whenever $P \overset{\alpha}{\Rightarrow} P'$ then $\alpha.P'$ is a summand of P. Now if P and Q are weakly congruent saturated normal form processes and $P \xrightarrow{\alpha} P'$ then $Q \overset{\alpha}{\Rightarrow} Q'$ for some Q' such that $Q' \approx P'$, where $\approx$ denotes weak bisimulation equivalence. By saturation, $Q \xrightarrow{\alpha} Q'$ and therefore $\alpha.Q'$ is a summand of Q. If, and this is a nontrivial if, we can deduce by the induction hypothesis that $\alpha.P'$ is provably equal to $\alpha.Q'$, which is much weaker than saying that P' is provably equal to Q', then we can conclude that every summand of P is provably equal to a summand of Q, and vice versa. This gives us the required completeness.

If one is only interested in a completeness proof, then the notion of saturated process is not needed. What is really necessary is the following *saturation property*, which is expressed by the *Absorption Lemma* in this paper.

If $P \overset{\alpha}{\Rightarrow} P'$ and P is in normal form, then P and $P + \alpha.P'$ are provably equal.

From the point of view of obtaining complete axiomatizations, the role of the saturation property is to relate operational semantics to equational rewriting. A careful examination of the role of the Hennessy Lemma in the completeness proof for CCS shows that what it really comes down to is the following property:

If $P \approx Q$ then either $\tau.P = Q$, or $P = Q$, or $P = \tau.Q$ is provable.

So the Hennessy Lemma helps to transfer a semantic statement to a proof theoretical one. As a matter of fact, as far as completeness is concerned, the following weaker property is all one needs:

If $P \approx Q$ then $\tau.P = \tau.Q$ is provable.

In this paper, following [10], we call this the *promotion property*, expressed by the Promotion Lemma (Lemma 9), which relates behavioral semantics to equational rewriting. It promotes a pair of semantically equivalent processes to a pair of proof theoretically equal processes. Just by the *Absorption Lemma* and the *Promotion Lemma*, we circumvent the Hennessy Lemma to obtain the completeness. It is fair to say that actually in [2], Aceto et al. provided a similar property as promotion lemma (see, [2], Proposition 4.3). However, there they focused on rooted branching bisimulation, and the motivation and details are quite different from ours.

Note that, as in [3] and unlike [2], we only focus on completeness rather than ω-completeness (i.e. completeness for equality of *open* terms over the signature of $\mathrm{BCCS}^{p*}(A_\tau)$). This is not a very serious shortcoming because of two reasons: (1) As in [3], using a technique due to Groote [11], it is not difficult to show ω-completeness. (2) We can provide the proof of ω-completeness by a minor modification on our proof, just as in [2]. We avoid doing this just because we intend to help the readers evade some unnecessary details and pay their full attention to the essence of our arguments.

The rest of the paper is organized as follows: Some preliminaries are reviewed in the following section. In Section 3, the simplified proof for completeness is presented. The paper is concluded with Section 4, where also related work is discussed.

2 Preliminaries

We assume a non-empty, countable set A of observable actions not containing the distinguished symbol τ. Following Milner, the symbol τ will be used to denote an internal, unobservable action of a system. We define $A_\tau \stackrel{\text{def}}{=} A \cup \{\tau\}$, and use a, b to range over A and α, β to range over A_τ. Note that we follow this convention strictly, so $a \neq \tau$ for any $a \in A$. We also assume a countably infinite set of process variables V, ranged over by x, y, z, that is disjoint from A_τ.

The language of basic CCS with prefix iteration, denoted by $\mathrm{BCCS}^{p*}(A_\tau)$, is given by the following BNF grammar:

$$P ::= x \mid 0 \mid \alpha.P \mid P + P \mid \alpha^* P$$

where $x \in V$ and $\alpha \in A_\tau$. The set of closed terms, i.e. terms that do not contain occurrences of process variables, generated by the above grammar, will be denoted by $\mathbf{T}(\mathrm{BCCS}^{p*}(A_\tau))$, while the set of open terms will be denoted by $\mathbb{T}(\mathrm{BCCS}^{p*}(A_\tau))$. We shall use P, Q (possibly subscripted and/or superscripted) to range over $\mathbb{T}(\mathrm{BCCS}^{p*}(A_\tau))$.

The operational semantics for the language $\mathrm{BCCS}^{p*}(A_\tau)$ is given by the labelled transition system $\langle \mathbf{T}(\mathrm{BCCS}^{p*}(A_\tau)), \{\stackrel{\alpha}{\rightarrow} \mid \alpha \in A_\tau\}\rangle$, where each transition relation $\stackrel{\alpha}{\rightarrow}$ is the least binary relation that satisfies the rules in **Fig**. 1. Following Milner [14], the derived transition relation $\Rightarrow$ is defined as the reflexive, transitive

$$\frac{}{\alpha.P \xrightarrow{\alpha} P} \quad \frac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'} \quad \frac{Q \xrightarrow{\alpha} Q'}{P + Q \xrightarrow{\alpha} Q'}$$

$$\frac{}{\alpha^* P \xrightarrow{\alpha} \alpha^* P} \quad \frac{P \xrightarrow{\alpha} P'}{\beta^* P \xrightarrow{\alpha} P'}$$

Fig. 1. Operational Semantics.

closure of $\xrightarrow{\tau}$, and $\overset{\alpha}{\Rightarrow}$, $\overset{\hat{\alpha}}{\Rightarrow}$ are defined in the standard way as follows:

$$\overset{\alpha}{\Rightarrow} \overset{def}{=} \Rightarrow\xrightarrow{\alpha}\Rightarrow$$
$$\overset{\hat{\alpha}}{\Rightarrow} \overset{def}{=} \begin{cases} \Rightarrow \text{if } \alpha = \tau \\ \overset{\alpha}{\Rightarrow} \text{otherwise.} \end{cases}$$

Now, we introduce some behavioral equivalences studied in this paper.

Definition 1. (Weak Bisimulation) A binary relation $\mathcal{R}$ over $\mathbf{T}(\text{BCCS}^{p*}(A_\tau))$ is a *weak bisimulation* if it is symmetric and whenever $P\mathcal{R}Q$ and $P \xrightarrow{\alpha} P'$, then there exists some Q' s.t. $Q \overset{\hat{\alpha}}{\Rightarrow} Q'$ and $P'\mathcal{R}Q'$.

Two process terms P, Q are observation equivalent, denoted by $P \approx Q$, if there exists a bisimulation $\mathcal{R}$ s.t. $P\mathcal{R}Q$.

As is well-known [14], $\approx$ is an equivalence relation. However, it is not a congruence w.r.t. the alternative composition operation and the prefix iteration operation.

Definition 2. (Observation Congruence) For all process terms $P, Q \in \mathbf{T}(\text{BCCS}^{p*}(A_\tau))$, $P \simeq Q$ iff for any $\alpha \in A_\tau$,

- If $P \xrightarrow{\alpha} P'$, then there exists some Q' s.t. $Q \overset{\alpha}{\Rightarrow} Q'$ and $P' \approx Q'$.
- If $Q \xrightarrow{\alpha} Q'$, then there exists some P' s.t. $P \overset{\alpha}{\Rightarrow} P'$ and $P' \approx Q'$.

Definition 3. For all $P, Q \in \mathbb{T}(\text{BCCS}^{p*}(A_\tau))$, $P \simeq Q$ iff $P\sigma \simeq Q\sigma$ if for every substitution $\sigma : V \to \text{BCCS}^{p*}(A_\tau)$.

Lemma 4. *The relation $\simeq$ is the largest congruence contained in $\approx$.*

Proof. cf. [2], Proposition 2.8.

Lemma 5. *Let $a, b \in A$. If $a^*P \approx b^*Q$, then $a = b$.*

Proof. cf. [2], Lemma 2.11.

3 Completeness of the Axiomatization

The main aim of this paper is to give a proof for completeness of the equational axiom system in [3][2] for observation congruence. We first present the axiom system. The axiom system $\mathcal{F}$ is the one that is shown in [6] to characterize strong bisimulation over $\mathbf{T}(\text{BCCS}^{p*}(A))$, which is given in **Fig**. 2. In addition, [3] extends it with two of Milner's standard τ-laws and three auxiliary equations that describe the interplay between the silent nature of τ and prefix iteration, which is reported in **Fig**. 3. We let $\mathcal{E}$ denote the system $\mathcal{F}$ together with these laws.

A1	$x + y = y + x$
A2	$(x + y) + z = x + (y + z)$
A3	$x + x = x$
A4	$x + 0 = x$
PA1	$a.(a^*x) + x = a^*x$
PB1	$a^*(a^*x) = a^*x$

Fig. 2. The Axiom System $\mathcal{F}$.

T1	$\alpha.\tau.x = \alpha.x$
T2	$\tau.x = \tau.x + x$
PT1	$\tau^*x = \tau.x$
PT2	$\tau.(a^*x) = a^*(\tau.(a^*x))$
AT3	$a.(x + \tau.y) = a.(x + \tau.y) + a.y$
PT3	$a^*(x + \tau.y) = a^*(x + \tau.y + a.y)$

Fig. 3. τ Laws for Observation Congruence.

For an axiom system $\mathcal{T}$, we write $\mathcal{T} \vdash P = Q$ iff the equation $P = Q$ is provable from the axiom system $\mathcal{T}$ using the rules of equational logic. Often for convenience, we omit $\mathcal{T}$ and abbreviate it as $\vdash P = Q$. We write $P \overset{X}{=} Q$ as a short-hand for $\text{A1}, \text{A2}, X \vdash P = Q$. We use $P =_{AC} Q$ to denote that P and Q are equal modulo associativity and commutativity of $+$, i.e. $\text{A1}, \text{A2} \vdash P = Q$.

For $I = \{i_1, \ldots, i_n\}$ a finite index set, we write $\sum_{i \in I} P_i$ for $P_{i_1} + \cdots + P_{i_n}$. By convention, if $I = \emptyset$, then $\sum_{i \in I} P_i$ stands for 0.

Soundness of the system is shown in [3]. The remainder of the section is devoted to an alternative proof of completeness w.r.t. [3]. As usual, we first identify a subset of process terms of a special form, which will be convenient in the proof of the completeness result for observation congruence. Following a long-established tradition of the literature on process theory, we shall refer to these terms as *normal forms*. The set of normal forms we are after is the smallest subset of process terms including process terms having one of the following two forms:

$$\sum_{i\in I}\alpha_i.P_i \quad \text{or} \quad a^*(\sum_{i\in I}\alpha_i.P_i)$$

where the term P_i are themselves normal forms, and I, J are finite index sets. (Recall that the empty sum represents 0).

Lemma 6. *Each term can be proven equal to a normal form using equations* A4*,* PA1*,* PB1*.*

Proof. cf. [2], Lemma 4.1. ☐

Note 7. PT1 is a powerful equation and is introduced in [4] under the name of "*Fair Iteration Rule*", it is an equational formulation of Koomen's *Fair Abstraction Rule* [5]. By using PT1, τ^*-like terms can be excluded from normal forms.

In the proof of the completeness result to come, we shall make use of a weight function $w : \mathbf{T}(\mathrm{BCCS}^{p*}(A_\tau)) \to \mathbb{N}$. This is defined by structured induction on terms as follows:

$$\begin{array}{llll} w(0) & = 0 & w(\alpha.P) & = w(P)+1 \\ w(P+Q) & = w(P)+w(Q)+1 & w(\alpha^*P) & = w(P)+1 \end{array}$$

Lemma 8. *(Absorption Lemma) For any $P, Q \in \mathbf{T}(\mathrm{BCCS}^{p*}(A_\tau))$, if $P \overset{\alpha}{\Rightarrow} P'$, then $\vdash P = P + \alpha.P'$.*

Proof. Standard result. See [14]. ☐

Lemma 9. *(Promotion Lemma) For all $P, Q \in \mathbf{T}(\mathrm{BCCS}^{p*}(A_\tau))$, if $P \approx Q$, then $\vdash \tau.P = \tau.Q$.*

Proof. By Lemma 6, it is sufficient to prove the statement of the lemma for weakly bisimilar normal forms P and Q. So let us assume that P and Q are weakly bisimilar normal forms and we show that $\tau.P = \tau.Q$ by induction on the sum of the weights of P and Q. Recall that normal forms can take the following two forms:

$$\sum_{i\in I}\alpha_i.P_i \quad \text{or} \quad a^*(\sum_{i\in I}\alpha_i.P_i)$$

where the P_i are themselves normal forms. So, in particular, P and Q have one of these forms. By symmetry, it is sufficient to deal with the following three cases:

1. $P = \sum_{i\in I}\alpha_i.P_i$ and $Q = \sum_{j\in J}\beta_j.Q_j$.
2. $P = a^*(\sum_{i\in I}\alpha_i.P_i)$ and $Q = b^*(\sum_{j\in J}\beta_j.Q_j)$.
3. $P = \sum_{i\in I}\alpha_i.P_i$ and $Q = a^*(\sum_{j\in J}\beta_j.Q_j)$.

We treat these three cases separately.

1. CASE: $P = \sum_{i \in I} \alpha_i.P_i$ and $Q = \sum_{j \in J} \beta_j.Q_j$. Consider a summand $\alpha_i.P_i$ of P. It gives rise to a transition $P \xrightarrow{\alpha_i} P_i$, and hence since $P \approx Q$, we can distinguish two subcases in the proof:
   - $\alpha_i \neq \tau$. Then we have $Q \stackrel{\alpha_i}{\Rightarrow} Q'$ and $P_i \approx Q'$. Since both P_i and Q' are in normal form and $w(P_i) + w(Q') < w(P) + w(Q)$, by the induction hypothesis, $\vdash \tau.P_i = \tau.Q'$. It follows that

   $$\begin{aligned} \vdash Q &= Q + \alpha_i.Q' && \text{(Lemma 8)} \\ &\stackrel{\text{T1}}{=} Q + \alpha_i.\tau.Q' \\ &= Q + \alpha_i.\tau.P_i \\ &\stackrel{\text{T1}}{=} Q + \alpha_i.P_i \end{aligned}$$

     Thus we can obtain that $\vdash Q = Q + \sum_{\alpha_i \neq \tau} \alpha_i.P_i$.
   - $\alpha_i = \tau$. Then either $Q \stackrel{\tau}{\Rightarrow} Q'$ or $Q = Q'$. In both cases, $P_i \approx Q'$. By the induction hypothesis, $\vdash \tau.P_i = \tau.Q'$. For the first case, it can be easily shown that $\vdash Q = Q + \tau.P_i$. In the second case one has $\vdash \tau.P_i = \tau.Q$.

   In summary, for $i \in I$, we have either $\tau.Q = \alpha_i.P_i$ or $Q = Q + \alpha_i.P_i$. It follows that $\vdash Q + P = Q + \sum_{\{i|\alpha_i \neq \tau\}} \alpha_i.P_i + \sum_{\{i|\alpha_i = \tau\}} \alpha_i.P_i = Q + \sum_{k \in K} \tau.Q$ for some index set $K = \{k \mid \alpha_k = \tau \text{ and } P_i \approx Q \text{ and } k \in I\}$. Consequently we have $\vdash \tau.(P+Q) = \tau.(Q + \sum_{k \in K} \tau.Q) \stackrel{\text{T2,A3}}{=} \tau.Q$. Symmetrically $\vdash \tau.(P+Q) = \tau.P$. Therefore $\vdash \tau.P = \tau.Q$.

2. CASE: $P = a^*(\sum_{i \in I} \alpha_i.P_i)$ and $Q = b^*(\sum_{j \in J} \beta_j.Q_j)$. First of all, note that by Lemma 5, it must be the case that $a = b$. For convenience, we write $P_1 = \sum_{i \in I} \alpha_i.P_i$ and $Q_1 = \sum_{j \in J} \beta_j.Q_j$. Then $P = a^* P_1$ and $Q = a^* Q_1$. For each transition $P \xrightarrow{\alpha_i} P_i$, since $P \approx Q$, we can distinguish three cases in the proof:
   - $P_1 \xrightarrow{\alpha_i} P_i$ and $\alpha_i \neq a, \tau$. Then it must be that $Q_1 \stackrel{\alpha_i}{\Rightarrow} Q'$ and $P_i \approx Q'$. By the induction hypothesis, $\vdash \tau.P_i = \tau.Q'$. Following the same lines as in CASE (1), we have $\vdash Q_1 = Q_1 + \alpha_i.P_i$. Thus we can obtain that $\vdash Q_1 = Q_1 + \sum_{\{i|\alpha_i \neq \tau, a\}} \alpha_i.P_i$.
   - $P_1 \xrightarrow{a} P_i$. Then there are three subcases:
     (i) $Q_1 \stackrel{a}{\Rightarrow} Q'$ and $P_i \approx Q'$. Then by the induction hypothesis, $\vdash \tau.P_i = \tau.Q'$. It follows that $\vdash Q_1 = Q_1 + a.P_i$;
     (ii) $Q \xrightarrow{a} Q$, $Q_1 \stackrel{\tau}{\Rightarrow} Q'$ and $P_i \approx Q'$. Then by the induction hypothesis, $\vdash \tau.P_i = \tau.Q'$. Moreover, we have $\vdash Q_1 = Q_1 + \tau.P_i$. Thus we can show that $\vdash Q_1 = Q_1 + \sum_{k \in K_1} \tau.P_k$ for some index set $K_1 \subseteq \{k \mid \alpha_k = a \text{ and } k \in I\}$;
     (iii) $Q \xrightarrow{a} Q$ and $P_i \approx Q$. Then by the induction hypothesis, $\vdash \tau.P_i = \tau.Q$. Thus $\vdash a.P_i = a.\tau.P_i = a.\tau.Q = a.Q$.

     Therefore, from (i)-(iii), we can conclude that $\vdash Q_1 + \sum_{k \in K_1 \cup K_2} a.P_k = Q_1 + \sum_{\{i|\alpha_i = a\}} a.P_i$ for the index set K_1 and some index set K_2, where $K_1, K_2 \subseteq \{k \mid \alpha_k = a \text{ and } k \in I\}$ and $K_1 \cap K_2 = \emptyset$. Moreover, $\vdash Q_1 =$

$Q_1 + \sum_{k\in K_1} \tau.P_k$ and $\vdash \tau.P_k = \tau.Q$ for $k \in K_2$. In the sequel, we write $K = K_1 \cup K_2$.

- $P_1 \xrightarrow{\tau} P_i$. Then either $Q_1 \overset{\tau}{\Rightarrow} Q'$ or $Q = Q'$ (note that $a \neq \tau$). In both cases, $P_i \approx Q'$. By the induction hypothesis, $\vdash \tau.P_i = \tau.Q'$. For the first case, it can be easily shown that $\vdash Q_1 = Q_1 + \tau.P_i$. In the second case one has $\vdash \tau.P_i = \tau.Q$. Thus $\vdash Q_1 + \sum_{l\in L} \tau.Q = Q_1 + \sum_{\{i \mid \alpha_i = \tau\}} \alpha_i.P_i$ for some index set $L = \{l \mid \alpha_l = \tau \text{ and } P_l \approx .Q \text{ and } l \in I\}$.

Combining the above three cases, actually we have $\vdash Q_1 + P_1 = Q_1 + \sum_{l\in L} \tau.Q + \sum_{k\in K_1\cup K_2} a.P_k$. Moreover

$$\vdash \; Q_1 \; = Q_1 + \sum_{k\in K_1} \tau.P_k \tag{1}$$

$$\vdash \tau.P_k = \tau.Q \quad \text{for } k \in K_2 \tag{2}$$

It follows that

$$\begin{aligned}
\vdash a^*(P_1 + Q_1) \;&=\; a^*(Q_1 + \sum_{l\in L} \tau.Q + \sum_{k\in K_1\cup K_2} a.P_k) \\
&\overset{(1)}{=} a^*(Q_1 + \sum_{l\in L} \tau.Q + \sum_{k\in K_1} a.P_k + \sum_{k\in K_2} a.P_k + \sum_{k\in K_1} \tau.P_k) \\
&\overset{\text{PT3}}{=} a^*(Q_1 + \sum_{l\in L} \tau.Q + \sum_{k\in K_1} \tau.P_k + \sum_{k\in K_2} a.P_k) \\
&\overset{(1)}{=} a^*(Q_1 + \sum_{l\in L} \tau.Q + \sum_{k\in K_2} a.P_k)
\end{aligned}$$

Now, we have to distinguish two cases:

- If $K_2 \neq \emptyset$, then we have

$$\begin{aligned}
\vdash a^*(P_1 + Q_1) \;&=\; a^*(Q_1 + \sum_{l\in L} \tau.Q + \sum_{k\in K_2} a.P_k) \\
&\overset{(2)}{=} a^*(Q_1 + \sum_{l\in L} \tau.Q + a.Q) \\
&\overset{\text{PA1}}{=} a^*(Q + \sum_{l\in L} \tau.Q)
\end{aligned}$$

  We proceed by considering the following two subcases:
  - $L = \emptyset$. Then $\vdash a^*(P_1 + Q_1) = a^*Q \overset{\text{PB1}}{=} Q$.
  - $L \neq \emptyset$. Then it follows that

$$\begin{aligned}
\vdash a^*(P_1 + Q_1) \;&=\; a^*(Q + \sum_{l\in L} \tau.Q) \\
&\overset{\text{T2}}{=} a^*(\tau.Q) \\
&= a^*(\tau.a^*Q_1) \\
&\overset{\text{PT2}}{=} \tau.Q
\end{aligned}$$

– If $K_2 \neq \emptyset$, then actually $\vdash a^*(P_1 + Q_1) = a^*(Q_1 + \sum_{l \in L} \tau.Q)$. Similarly, we also proceed by considering the following two subcases:
  - $L = \emptyset$. Then $\vdash a^*(P_1 + Q_1) = a^*Q_1 = Q$.
  - $L \neq \emptyset$. Then it follows that

$$\begin{aligned}
\vdash a^*(P_1 + Q_1) &= a^*(Q_1 + \sum_{l \in L} \tau.Q) \\
&\overset{\text{A3,T2}}{=} a^*(Q_1 + \tau.Q + a^*Q_1) \\
&\overset{\text{PA1}}{=} a^*(Q_1 + \tau.Q + a.a^*Q_1 + Q_1) \\
&\overset{\text{A3,PA1}}{=} a^*(\tau.Q + Q) \\
&\overset{\text{T2}}{=} a^*(\tau.a^*Q_1) \\
&\overset{\text{PT2}}{=} \tau.Q
\end{aligned}$$

From the above, we conclude that either $\vdash \tau.Q = a^*(P_1 + Q_1)$ or $\vdash Q = a^*(P_1 + Q_1)$. Symmetrically $\vdash \tau.P = a^*(P_1 + Q_1)$ or $P = a^*(P_1 + Q_1)$. For each of four combinations, clearly, we have $\vdash \tau.P = \tau.Q$, possibly using T1.

3. CASE: $P = \sum_{i \in I} \alpha_i.P_i$ and $Q = a^*(\sum_{j \in J} \beta_j.Q_j)$. For convenience, we write $Q_1 = \sum_{j \in j} \beta_j.Q_j$. We proceed by examining to the two directions of observation congruence.
   – For each transition from P, we consider three subcases in the proof:
     - $P \xrightarrow{\alpha_i} P_i$ and $\alpha_i \neq \tau, a$. Since $P \approx Q$, this transition from P must be matched by a transition $Q_1 \overset{\alpha_i}{\Rightarrow} Q'$ and $P_i \approx Q'$.
     - $P \xrightarrow{a} P_i$. Since $P \approx Q$, this transition from P must be matched by transitions $Q \xrightarrow{a} Q$, $Q_1 \overset{\tau}{\Rightarrow} Q'$ and $P_i \approx Q'$, or $Q \xrightarrow{a} Q$ and $P_i \approx Q$, or $Q_1 \overset{a}{\Rightarrow} Q'$ and $P_i \approx Q'$.
     - $P \xrightarrow{\tau} P_i$. Since $P \approx Q$, this transition from P must be matched by transitions $Q_1 \overset{\tau}{\Rightarrow} Q'$ or just $Q = Q'$. In both cases, $P_i \approx Q'$.

   Following the same lines of CASE (2), we can conclude that either $\vdash \tau.Q = a^*(P + Q_1)$ or $\vdash Q = a^*(P + Q_1)$.
   – For the transition from Q, we only need to consider the following two situations:
     - $Q_1 \xrightarrow{\beta_j} Q_j$. Since $P \approx Q$, it must be matched by a transition $P \overset{\hat{\beta}_j}{\Rightarrow} P'$ and $P' \approx Q_j$. Following the same line of CASE (1), we obtain that $\vdash \tau.(P + Q_1) = \tau.P$.
     - $Q \xrightarrow{a} Q$. Since $P \approx Q$, it must be matched by a transition $P \overset{a}{\Rightarrow} P'$ and $P' \approx Q$. By the induction hypothesis, we have $\vdash \tau.P' = \tau.Q$. Clearly, it follows that $\vdash P = P + a.Q$.

   Then because $\vdash P = P + a.Q = P + a.a^*(P + Q_1)$ or $\vdash P = P + a.Q = P + a.\tau.Q = P + a.a^*(P + Q_1)$, we have $\vdash P + Q_1 = P + Q_1 + a.a^*(P + Q_1) \overset{\text{PA1}}{=} a^*(P + Q_1)$. It follows that $\vdash \tau.Q = \tau.(P + Q_1) = \tau.P$.

The proof is complete. □

Theorem 10. *(Completeness) If $P \simeq Q$, then $\vdash P = Q$.*

Proof. Consider two process terms P and Q that are observation congruent. We shall show that $P + Q$ is provably equal to Q. Of course, by symmetry, $P + Q$ is also provably equal to P. Hence we obtain completeness.

To this end, note that by Lemma 6, P and Q may be proven equal to some normal forms using A4, PA1 and PB1. Possibly using equation PA1 again, we may therefore derive that $\vdash P = \sum_{i \in I} \alpha_i.P_i$ and $\vdash Q = \sum_{j \in J} \beta_j.Q_j$ for some finite index sets I, J. Consider a summand $\alpha_i.P_i$ of P. It gives rise to a transition $P \stackrel{\alpha_i}{\to} P_i$ and hence, since $P \simeq Q$, there exists some Q' such that $Q \stackrel{\alpha_i}{\Rightarrow} Q'$ and $P_i \approx Q'$. By Lemma 9, we have $\vdash \tau.P_i = \tau.Q'$. It follows that

$$\begin{aligned} \vdash Q &= Q + \alpha_i.Q' && \text{(Lemma 8)} \\ &\stackrel{\mathrm{T1}}{=} Q + \alpha_i.\tau.Q' \\ &= Q + \alpha_i.\tau.P_i \\ &\stackrel{\mathrm{T1}}{=} Q + \alpha_i.P_i \end{aligned}$$

Consequently, we have $\vdash Q = Q + \sum_{i \in I} \alpha_i.P_i = Q + P$. By symmetry, $\vdash P = P + Q$. Thus $\vdash P = P + Q = Q$. The proof is complete. $\square$

4 Conclusion

Related Work. Besides the works we have pointed out in Section 1, we would like to mention some other related works, which are mainly dealing with the axiomatization of strong bisimilarity over BPA with Kleene star-like operation. In [15], Milner first studies iteration in strong bisimulation equivalence in a process algebra equivalent to $\mathrm{BPA}_{\delta\epsilon}$ extended with the Kleene star. Bergstra, Bethke and Ponse [4] consider BPA with the Kleene star, and they suggest a finite equational axiomatization for this algebra. In [8], Fokkink and Zantema prove that this axiomatization is complete w.r.t. strong bisimulation equivalence. In [7], Fokkink presents a simpler and shorter completeness proof. In [9], Fokkink and Zantema also study a rewrite system that stems from axioms for prefix iteration in $\mathrm{BPA}_{\delta\epsilon}$ and obtain a complete axiomatization.

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