Problems 8: The Black-Scholes theory

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Question 1

The following is the Black-Scholes equation describing how the value $V(S,t)$ of an option depends on the underlying stock price S and time t, when stock is paying continuous dividend at rate ρ:

$$\frac{\partial V}{\partial t} = rV - (r-\rho)S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

Which part of this equation is usually denoted by the Greek letter Δ? What does it represent? How is it used for Δ-hedging? If V is a call option, what is the value of Δ, if S is significantly below the strike price?

Answer: *Δ is the first partial derivative $\partial V/\partial S$. It represents the change of the option's value relative to changes in the underlying stock. Δ-hedging is used to replicate the changes in the value of option by buying (possibly negative) amount Δ of stock. When a call is far out of the money, the value function V is almost constant (close to zero), and so Δ is close to zero.*

Question 2

Consider the Black-Scholes equation in the previous question. Which part of this equation is usually denoted by the Greek letter Γ? What does it represent? How is it used for Γ-hedging? What is the value of Γ, if S is far away from the strike price?

Answer: *Γ is the second partial derivative $\partial^2 V/\partial S^2$. It represents the curvature (convexity for long and concavity for short positions) of the option's value relative to changes in the underlying stock. When Γ is large, and there is a significant change of S, then the first derivative $\partial V/\partial S$ does not predict well the optimal amount of stock to buy or sell used for Δ-hedging, which is known as Γ-risk. Thus, Γ-hedging is used to minimise Γ-risk by replicate the curvature by buying (possibly negative) amount of another option. When stock S is far from the strike price, the value function V has almost zero curvature, and so Γ is close to zero.*

Question 3

The following equations are the Black-Scholes prices of call and put options with stock S paying dividends at constant continuous rate ρ:

$$\begin{aligned} C(S,t) &= e^{-\rho(T-t)}SN(d_1) - e^{-r(T-t)}KN(d_2) \\ P(S,t) &= e^{-r(T-t)}KN(-d_2) - e^{-\rho(T-t)}SN(-d_1) \end{aligned}$$

where

$$d_1 = \frac{\ln(S/K) + (r-\rho+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

Differentiate over S to derive the equations for Δ and Γ. Hint: use the fact that the CDF of the standard normal distribution ($\mu = 0$, $\sigma^2 = 1$) is $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-s^2/2}\,ds$, and its derivative is $dN(x)/dx = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

Answer: *For* Δ*s we have:*

$$\begin{aligned} \frac{\partial}{\partial S}C(S,t) &= e^{-\rho(T-t)}N(d_1) + e^{-\rho(T-t)}S\frac{\partial}{\partial S}N(d_1) - e^{-r(T-t)}K\frac{\partial}{\partial S}N(d_2) \\ \frac{\partial}{\partial S}P(S,t) &= -e^{-\rho(T-t)}N(-d_1) + e^{-\rho(T-t)}S\frac{\partial}{\partial S}N(d_1) - e^{-r(T-t)}K\frac{\partial}{\partial S}N(d_2) \end{aligned}$$

We shall now prove that

$$e^{-\rho(T-t)}S\frac{\partial}{\partial S}N(d_1) - e^{-r(T-t)}K\frac{\partial}{\partial S}N(d_2) = 0 \tag{1}$$

Consider the derivatives:

$$\begin{aligned} \frac{\partial}{\partial S}N(d_1) &= \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}\frac{1}{S\sigma\sqrt{T-t}} \\ \frac{\partial}{\partial S}N(d_2) &= \frac{1}{\sqrt{2\pi}}e^{-d_2^2/2}\frac{1}{S\sigma\sqrt{T-t}} \end{aligned}$$

where we used the fact that $\partial d_1(S)/\partial S = \partial d_2(S)/\partial S = \frac{1}{S\sigma\sqrt{T-t}}$, *because* $d_2 = d_1 - \sigma\sqrt{T-t}$. *The first equation gives*

$$e^{-\rho(T-t)}S\frac{\partial}{\partial S}N(d_1) = e^{-\rho(T-t)}\frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi(T-t)}} \tag{2}$$

Using $d_2 = d_1 - \sigma\sqrt{T-t}$ *and the definition of* d_1 *obtain:*

$$\begin{aligned} -\frac{1}{2}d_2^2 &= -\frac{1}{2}d_1^2 + d_1\sigma\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t) \\ &= -\frac{1}{2}d_1^2 + \ln\frac{S}{K} + (r-\rho)(T-t) \end{aligned}$$

This allows us to rewrite $e^{-d_2^2/2}$ *as follows*

$$e^{-d_2^2/2} = e^{r(T-t)} e^{-\rho(T-t)} \frac{S}{K} e^{-d_1^2/2}$$

Therefore

$$e^{-r(T-t)} K \frac{\partial}{\partial S} N(d_2) = e^{-\rho(T-t)} \frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi(T-t)}} \tag{3}$$

One can see the equality of the right-hand-sides of equations (2) and (3), which proves our assertion (1). The final expressions for Δ*s are*

$$\begin{aligned} \frac{\partial}{\partial S} C(S,t) &= e^{-\rho(T-t)} N(d_1) \\ \frac{\partial}{\partial S} P(S,t) &= -e^{-\rho(T-t)} N(-d_1) \end{aligned}$$

For Γ*s we have:*

$$\begin{aligned} \frac{\partial^2}{\partial S^2} C(S,t) &= e^{-\rho(T-t)} \frac{\partial}{\partial S} N(d_1) = e^{-\rho(T-t)} \frac{e^{-d_1^2/2}}{S\sigma\sqrt{2\pi(T-t)}} \\ \frac{\partial^2}{\partial S^2} P(S,t) &= -e^{-\rho(T-t)} \frac{\partial}{\partial S} N(-d_1) = e^{-\rho(T-t)} \frac{e^{-d_1^2/2}}{S\sigma\sqrt{2\pi(T-t)}} \end{aligned}$$

Note that Γ*s are the same for calls and puts.*

Question 4

Check that the Black-Scholes prices for call and put satisfy the call-put parity: $C - P = e^{-\rho(T-t)} S - e^{-r(T-t)K}$. Hint: use the fact that $N(x) = 1 - N(-x)$ for the CDF of normal distribution.

Answer: *Subtract* $C - P$ *and use* $N(x) + N(-x) = N(x) + 1 - N(x) = 1$*:*

$$\begin{aligned} & C - P \\ &= e^{-\rho(T-t)} S[N(d_1) + N(-d_1)] - e^{-r(T-t)} K[N(d_2) + N(-d_2)] \\ &= e^{-\rho(T-t)} S - e^{-r(T-t)} K \end{aligned}$$