Problems 7:
Ito differentiation rule

Roman Belavkin

Middlesex University

Question 1

Consider the Ito formula for the differential of function $F(x,t)$:

$$dF(x,t) = \dot{F}\,dt + F'\,dx(t) + \frac{1}{2}F''\,dx^2(t)$$

What is the formula for dF in case of a differentiable process $x(t)$? (i.e. a process, for which $\dot{x}(t) = dx(t)/dt$ exists).

Answer: *For differentiable $x(t)$, its differential is $dx(t) = \dot{x}\,dt$, and $dx^2 = (\dot{x}\,dt)^2 = \dot{x}^2\,dt^2 = \dot{x}^2 \cdot 0 = 0$, because $dt^2 = 0$. The differential then has the form:*

$$dF(x,t) = \dot{F}\,dt + F'\,dx(t)$$

Question 2

Obtain differentials $dy = dF(x,t)$ for the following functions:

a) $y = \frac{x^2}{2} + e^{-at}$

b) $y = \ln x$

c) $y = e^x$

d) $y = \sin(x)$

Answer:

a) $\dot{F} = -ae^{-at}$, $F' = x$, $F'' = 1$:

$$dy = \dot{F}\,dt + F'\,dx + \frac{1}{2}F''\,dx^2 = -ae^{-at}\,dt + x\,dx + \frac{1}{2}\,dx^2$$

b) $\dot{F} = 0$, $F' = 1/x$, $F'' = -1/x^2$:

$$dy = \frac{dx}{x} - \frac{1}{2}\frac{dx^2}{x^2}$$

c) $\dot{F} = 0$, $F' = e^x$, $F'' = e^x$:

$$dy = e^x(dx + \frac{1}{2}\, dx^2)$$

d) $\dot{F} = 0$, $F' = \cos(x)$, $F'' = -\sin(x)$:

$$dy = \cos(x)\, dx - \frac{1}{2}\sin(x)\, dx^2$$

Question 3

Complete the expressions for the differentials dy from the previous question for the process $x(t)$ described by the following stochastic differential equation:

a) $dx = \sqrt{2a}e^{-at/2}\, dw$

b) $\frac{dx}{x} = \mu\, dt + \sigma\, dw$

c) $dx = \mu x\, dt + \sigma\, dw$

d) $dx = \cos(x)\, dt + \sqrt{\frac{2}{\sin(x)}}\, dw$

Answer: *For* $dx = f(x,t)\, dt + g(x,t)\, dw$, *use the formula* $dx^2 = g^2(x,t)\, dt$:

a)

$$dy = -ae^{-at}\, dt + x\sqrt{2a}e^{-at/2}\, dw + ae^{-at}\, dt = x\sqrt{2a}e^{-at/2}\, dw$$

b)

$$dy = \mu\, dt + \sigma\, dw(t) - \frac{1}{2}(\mu\, dt + \sigma\, dw)^2 = \left(\mu - \frac{1}{2}\sigma^2\right)\, dt + \sigma\, dw$$

c)

$$dy = e^x\left[\left(\mu x + \frac{1}{2}\sigma^2\right)\, dt + \sigma\, dw\right]$$

d)

$$dy = \cos^2(x)\, dt + \cos(x)\sqrt{\frac{2}{\sin(x)}}\, dw - dt = (\cos^2(x) - 1)\, dt + \cos(x)\sqrt{\frac{2}{\sin(x)}}\, dw$$

Question 4

Is any of the processes $y(t)$, obtained in the previous question, differentiable at any t?

Answer: *No, as each of the processes includes stochastic differential dw, which is nowhere differentiable.*