Problems 5: Continuous Markov process and the diffusion equation

Roman Belavkin

Middlesex University

Question 1

Give a definition of Markov stochastic process. What is a continuous Markov process?

Answer: *A process $x(t)$ is Markov, if for any $t_1, \dots, t_n, t_{n+1}$ the conditional probability density has the property:*

$$p(x(t_{n+1}) \mid x(t_n), \dots, x(t_1)) = p(x(t_{n+1}) \mid x(t_n))$$

This is sometimes referred as a process with no memory or with no aftereffect.

Markov process is called continuous, *if for all moments $\mathbb{E}\{(\Delta x(t))^n \mid x(t)\} = 0$ with $n > 2$ the following coefficients tend to zero:*

$$K_n(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}\{(\Delta x(t))^n \mid x(t)\} = 0 \,, \qquad \forall\, n > 2$$

The first and the second coefficients are called drift *and* diffusion coefficients*:*

$$K_1(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}\{\Delta x(t) \mid x(t)\} \,, \quad K_2(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}\{(\Delta x(t))^2 \mid x(t)\}$$

Continuous Markov processes are described by the Fokker-Planck equation:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[K_1(x,t)p(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[K_2(x,t)p(x,t)]$$

Question 2

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of independent identically distributed random variables. Which of the following processes is Markov?

a) $y_n = x_1 + \cdots + x_n$

b) $y_n = x_1 \times \cdots \times x_n$

c) $y_n = \max\{0, x_1, \ldots, x_n\}$

d) $y_n = \left(n, \frac{x_1+\cdots+x_n}{n}\right)$

Hint: convert to iteration $y_{n+1} = f(y_n)$.

Answer: *All of the processes* $\{y_n\}_{n\in\mathbb{N}}$ *are Markov, as they can be represented by the following iterations:*

a) $y_{n+1} = y_n + x_{n+1}$, $y_1 = 0$.

b) $y_{n+1} = y_n \times x_{n+1}$, $y_1 = 1$.

c) $y_{n+1} = \max\{y_n, x_{n+1}\}$, $y_1 = 0$.

d) $y_{n+1} = (z_{n+1}, v_{n+1}) = \left(z_n + 1, \left(\frac{v_n + x_{n+1}}{z_n + 1}\right)\right)$, $y_1 = (1, x_1)$.

Question 3

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of independent identically distributed random variables. Why is the following process not Markov?

$$y_n = [x_1 + \cdots + x_n]$$

where $[\cdot]$ means rational part.

Answer: *Although the sum of independent random variables is Markov (see previous example), the rational part* y_{n+1} *of the sum* $x_1 + \cdots + x_n$ *depends on all elements of the sequence, not just* x_{n+1}*. Therefore,* y_{n+1} *cannot be computed from* y_n *and* x_{n+1}*.*

Question 4

Let $\{m(t)\}_{t\geq 0}$ be a stochastic process defined by

$$m(t) = \frac{n(t) - \nu t}{\sqrt{\nu}}$$

where $n(t)$ is the value of a stationary Poisson process $\{n(t)\}_{t\geq 0}$ with expected value $\mathbb{E}\{n(t)\} = \nu t$ (see Appendix A). Use properties of the Poisson process to

a) Show that the expected value and variance of $m(t)$ are:

$$\mathbb{E}\{m(t)\} = 0\,, \qquad \sigma^2(m(t)) = t$$

Answer: *Expected value:*

$$\mathbb{E}\{m(t)\} = \mathbb{E}\left\{\frac{n(t) - \nu t}{\sqrt{\nu}}\right\} = \frac{\mathbb{E}\{n(t)\} - \nu t}{\sqrt{\nu}} = \frac{\nu t - \nu t}{\sqrt{\nu}} = 0$$

Variance:

$$\begin{aligned}
\sigma^2(m(t) &= \mathbb{E}\{m^2(t)\} - (\mathbb{E}\{m(t)\})^2 = \mathbb{E}\{m^2(t)\} \quad \textit{(by } \mathbb{E}\{m(t)\} = 0\textit{)} \\
&= \mathbb{E}\left\{\left[\frac{n(t) - \nu t}{\sqrt{\nu}}\right]^2\right\} \\
&= \mathbb{E}\left\{\frac{n^2(t) - 2n(t)\nu t + (\nu t)^2}{\nu}\right\} \\
&= \frac{\mathbb{E}\{n^2(t)\} - 2\mathbb{E}\{n(t)\}\nu t + (\nu t)^2}{\nu} \\
&= \frac{\mathbb{E}\{n^2(t)\} - 2(\nu t)^2 + (\nu t)^2}{\nu} && \textit{(by } \mathbb{E}\{n(t)\} = \nu t\textit{)} \\
&= \frac{\mathbb{E}\{n^2(t)\} - (\nu t)^2}{\nu} \\
&= \frac{\mathbb{E}\{n^2(t)\} - \mathbb{E}\{n(t)\}^2}{\nu} && \textit{(by } \mathbb{E}\{n(t)\} = \nu t\textit{)} \\
&= \frac{\nu t}{\nu} = t && \textit{(by } \sigma^2(n(t)) = \nu t\textit{)}
\end{aligned}$$

b) Prove that the differential $dm(t) = m(t + dt) - m(t)$ has the property

$$\lim_{\nu \to \infty} (dm^2) = dt$$

Answer: *The differential dm is*

$$dm(t) = m(t+dt) - m(t) = \frac{n(t + dt) - \nu(t + dt)}{\sqrt{\nu}} - \frac{n(t) - \nu t}{\sqrt{\nu}} = \frac{dn(t) - \nu dt}{\sqrt{\nu}}$$

The formula for dm^2 is

$$\begin{aligned}
dm^2 &= \frac{(dn - \nu dt)^2}{\nu} \\
&= \frac{dn^2 - 2dn\nu dt + \nu^2 dt^2}{\nu} \\
&= \frac{dn^2}{\nu} && \text{(by } dn = 0 \text{ a.e. and } dt^2 = 0\text{)} \\
&= \frac{dn}{\nu} && \text{(by } dn^2 = dn\text{)} \\
&= \frac{dn}{\nu} - dt + dt \\
&= \frac{1}{\sqrt{\nu}} \left(\frac{dn - \nu dt}{\sqrt{\nu}} \right) + dt \\
&= \frac{1}{\sqrt{\nu}} dm + dt
\end{aligned}$$

Using $dm^2 = \frac{dm}{\sqrt{\nu}} + dt$

$$\lim_{\nu \to \infty} (dm^2) = \lim_{\nu \to \infty} \left(\frac{dm}{\sqrt{\nu}} + dt \right) = dt$$

c) The stochastic process $\{m(t)\}_{t \geq 0}$ becomes Wiener process as $\nu \to \infty$.

Answer: *Recall that Wiener process $\{w(t)\}_{t \geq 0}$ is a stationary Gaussian stochastic process with independent increments $\Delta w(t)$ and with the expected value and variance equal to*

$$\mathbb{E}\{w(t)\} = 0\,, \qquad \sigma^2(w(t)) = t$$

and with stochastic differentials $dw(t)$ having the property:

$$dw^2 = dt$$

We have shown that the expected value and variance of stochastic process $\{m(t)\}_{t \geq 0}$ are also equal to

$$\mathbb{E}\{m(t)\} = 0\,, \qquad \sigma^2(m(t)) = t$$

The increments

$$\Delta m(t) = \frac{\Delta n(t) - \nu \Delta t}{\sqrt{\nu}}$$

are independent, because $\Delta n(t)$ are independent increments of the Poisson process.

We have shown also that as the rate of events increases $\nu \to \infty$, the differentials dm^2 tend to dt, which agrees with the property $dw^2 = dt$ of the Wiener process.

It only remains to show that the probability distribution of $m(t)$ tends to Gaussian distribution as $\nu \to \infty$. This follows form the fact that $m(t)$ is a transformation of the values $n(t)$ of a Poisson process, so that the distribution of $m(t)$ is a push-forward probability $P(n(t) = \sqrt{\nu}m(t) + \nu t)$, where $P(n(t))$ is the Poisson distribution with the expected rate ν. It is well-known that the Poisson distribution with large rate ν (and hence large $\mathbb{E}\{n(t)\} = \nu t$) can be approximated by Gaussian distribution with the mean and variance equal to ν. This can be understood from the fact that both the Poisson and the Gaussian distributions can be obtained from binomial distribution in the limit $n \to \infty$ (these facts are known as the Poisson theorem and the De Moivre-Laplace theorems respectively). Therefore, in the limit $\nu \to \infty$, stochastic process $m(t)$ becomes Gaussian with expected value $\mathbb{E}\{m(t)\} = 0$ and variance $\sigma^2(m(t)) = t$.

A Poisson process

The Poisson point process $\{n(t)\}_{t\geq 0}$ is a discrete-valued stochastic process counting the number $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ of occurrences of some event during time interval $[0, t]$, and satisfying the following properties:

Independent increments : The number $\Delta n(\Delta t) = n(t + \Delta t) - n(t)$ of events in the interval Δt is independent of the number of events in any other interval non-overlapping with Δt (e.g. $[0, t]$). This property implies $\{n(t)\}_{t\geq 0}$ is a Markov process.

Orderliness : The probability of two or more events during sufficiently small interval Δt is essentially zero:

$$P\{\Delta n(\Delta t) \geq 2\} = o(\Delta t)$$

(note the use of the small 'o' notation.)

These two properties imply that the number $n(t)$ of events in $[0, t]$ has Poisson distribution. For stationary (or homogeneous) Poisson process this distribution is

$$P(n(t)) = \frac{(\nu t)^n}{n!}\, e^{-\nu t}$$

where ν is the expected *rate* or *intensity* parameter (the expected number of events in a unit interval). The expected value and the variance for stationary Poisson process are respectively

$$\mathbb{E}\{n(t)\} = \nu t\,, \qquad \sigma^2(n(t)) = \nu t$$

The orderliness property implies also that differential $dn(t) = n(t+dt) - n(t)$ can have only two values: $dn(t) = 0$ (almost everywhere) or $dn(t) = 1$ (in a set of measure zero). This means that the differential dn of a Poisson process satisfies the property

$$dn^2 = dn$$