Problems 4:
Gaussian white noise and Wiener process

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Question 1

Compute characteristic function $\theta(u)$ of the uniform density function $p(x) = \frac{1}{b-a}$ if $x \in [a, b]$, and $p(x) = 0$ otherwise. Hint: use the inverse Fourier transform

$$\theta(u) = \mathcal{F}[p(x)](u) := \int_{-\infty}^{\infty} e^{iux}\, p(x)\, dx$$

Answer: *Because the function $p(x)$ is zero outside the interval $[a, b]$, we only need take the integral from a to b:*

$$\theta(u) = \int_a^b e^{iux} \frac{1}{b-a}\, dx = \frac{1}{b-a} \frac{1}{iu}\, e^{iux}\Big|_a^b = \frac{e^{iub} - e^{iua}}{iu(b-a)}$$

Question 2

The characteristic function of Gaussian density $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is

$$\theta(u) = \int_{-\infty}^{\infty} e^{iux} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = e^{iu\mu - \frac{\sigma^2}{2} u^2}$$

Compute all derivatives of the cumulant generating function $\Gamma(u) = \ln \theta(u)$.

Answer: *The cumulant generating function is $\Gamma(u) = iu\mu - \frac{\sigma^2}{2} u^2$. Its derivatives are:*

$$\Gamma'(u) = i\mu - \sigma^2 u\,, \qquad \Gamma''(u) = -\sigma^2 = i^2\sigma^2\,, \qquad \Gamma^{(n)}(u) = 0\,, \quad \forall\, n > 2$$

Question 3

Use the property $f(0) = \int f(x)\delta(x)\, dx$ of the Dirac δ-function to show that its Fourier transform is a constant function $\mathcal{F}[\delta(x)](y) = 1$.

Answer: *Applying the Fourier transform $\hat{g}(y) = \int g(x)e^{-ixy}\,dx$ to $g(x) = \delta(x)$, and using the property of δ-function with $f(x) = e^{-ixy}$, we have:*

$$\mathcal{F}[\delta(x)](y) = \int_{-\infty}^{\infty} \delta(x)e^{-ixy}\,dx = e^{-i0y} = 1$$

Question 4

The Dirac δ-function can be approximated by a Gaussian density function with small variance:

$$\delta(x) = \lim_{\sigma\to 0}\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}\right)$$

Use this property to prove that Fourier transform of $\delta(x)$ is a constant function.

Answer: *Fourier transform of the Gaussian distribution is*

$$\mathcal{F}[p(x)](t) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}-ixt}\,dx = \frac{\sigma\sqrt{2\pi}}{\sigma\sqrt{2\pi}}e^{-\frac{t^2\sigma^2}{2}} = e^{-\frac{t^2\sigma^2}{2}}$$

Taking the limit under the integral and moving it outside the integral gives:

$$\mathcal{F}[\lim_{\sigma\to 0} p(x)](t) = \lim_{\sigma\to 0}\mathcal{F}[p(x)](t) = \lim_{\sigma\to 0} e^{-\frac{t^2\sigma^2}{2}} = 1\,, \quad \forall\, t \in \mathbb{R}$$

Question 5

Prove that a δ-correlated process has constant spectral density, and therefore an unbounded (i.e. infinite) variance σ^2. Hint: use the definition of the spektral density as the Fourier transform of the correlation function (in this case $k(\tau) = K\delta(\tau)$, where K is a constant), and the fact that $\sigma^2 = k(0)$.

Answer: *The spektral density of a δ-correlated process is:*

$$S(\lambda) = \int k(\tau)e^{-i\tau\lambda}\,d\tau = \int K\delta(\tau)e^{-i\tau\lambda}\,d\tau = K\int \delta(\tau)e^{-i\tau\lambda}\,d\tau = K\cdot 1$$

The auto-correlation function $k(\tau)$ is the inverse Fourier transform of the spektral density:

$$k(\tau) = \frac{1}{2\pi}\int S(\lambda)e^{i\lambda\tau}\,d\lambda$$

The fact that the variance is unbounded follows from the property $\sigma^2 = k(0)$ and the fact that $S(\lambda)$ is a constant:

$$\sigma^2 = k(0) = \frac{1}{2\pi}\int S(\lambda)e^{i\lambda 0}\,d\lambda = \frac{1}{2\pi}\int K\,d\lambda \to \infty$$

Question 6

Consider the following probability density function for the values x_1 and x_2 of a stochastic process at two moments in time:

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - R_{12}^2}} e^{-\frac{1}{2(1-R_{12}^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} + 2R_{12}\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]}$$

Prove that x_1 and x_2 are independent if and only if they have zero correlation $R_{12} = 0$.

Answer: *The joint density $p(x_1, x_2)$ is a product of two Gaussian densities if and only if $R_{12} = 0$:*

$$p(x_1, x_2) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x_1-\mu_1)^2}{\sigma_1^2}} \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x_2-\mu_2)^2}{\sigma_2^2}}$$

Question 7

Give the definitions of a Gaussian white noise, standard white noise and the Wiener process.

Answer: *Gaussian white noise is a stationary stochastic process with autocorrelation function $k(\tau) = N\delta(\tau)$ (i.e. δ-correlated Gaussian process). It is called standard white noise if it has zero mean and $N = 1$, that is:*

$$\mathbb{E}\{\xi(t)\} = 0\,, \quad \mathbb{E}\{\xi(t)\xi(t+\tau)\} = k(\tau) = \delta(\tau)$$

The Wiener process is the process $w(t)$ with independent increments $\Delta w(t) = w(t + \Delta t) - w(t)$, which have Gaussian distribution with zero mean and variance $\mathbb{E}\{(w(t + \Delta t) - w(t))^2\} = \Delta t$. Because the increments $\Delta w(t)$ are independent, they are δ-correlated, and therefore they represent a Gaussian white noise.

Question 8

Prove that the Wiener process $w(t)$ is nowhere differentiable in probability (i.e. the probability that time derivative of $w(t)$ exists for some t is zero). Hint: Use the definition of a derivative as the limit of the quotient $\Delta w/\Delta t$ for $\Delta t \to 0$, and the fact that the variance $\mathbb{E}\{(\Delta w)^2\}$ of the increments $\Delta w = w(t + \Delta t) - w(t)$ of the Wiener process $w(t)$ is Δt.

Answer: *Recall the definition of the derivative of $w(t)$:*

$$\frac{dw(t)}{dt} = \lim_{\Delta t \to 0} \frac{w(t + \Delta t) - w(t)}{\Delta t}$$

Because $w(t)$ is the Wiener process, $\Delta w = w(t + \Delta t) - w(t)$ has Gaussian distribution with zero mean and variance Δt. This means that the quotient $\Delta w(t)/\Delta t$ has Gaussian distribution with zero mean and variance $1/\Delta t$:

$$\mathbb{E}\left\{\frac{(\Delta w(t))^2}{\Delta t^2}\right\} = \frac{\Delta t}{\Delta t^2} = \frac{1}{\Delta t}$$

The above value does not have a limit at $\Delta t \to 0$. Therefore, as $\Delta t \to 0$, the probability that the quotient is less than some λ converges to zero:

$$\lim_{\Delta t \to 0} P\left\{\frac{\Delta w(t)}{\Delta t} < \lambda\right\} = 0$$

Thus, $w(t)$ is not differentiable at any t in probability.

Question 9

What is the spektral density of a stationary process with auto-correlation $k(\tau) = \sigma^2 e^{-\beta|\tau|}$ with $\sigma^2 = 1$ (i.e. Gaussian exponentially correlated process)? For which values of β can we model such a process by a standard white noise?

Answer: *The spektral density is the Fourier transform of $k(\tau)$:*

$$\begin{aligned}
S(\lambda) &= \int_{-\infty}^{\infty} e^{-\beta|\tau|} e^{-i\tau\lambda}\, d\tau \\
&= \int_{-\infty}^{0} e^{\tau(\beta - i\lambda)}\, d\tau + \int_{0}^{\infty} e^{-\tau(\beta + i\lambda)}\, d\tau \\
&= \left.\frac{e^{\tau(\beta - i\lambda)}}{\beta - i\lambda}\right|_{-\infty}^{0} + \left.\frac{e^{-\tau(\beta + i\lambda)}}{\beta + i\lambda}\right|_{0}^{\infty} \\
&= \frac{1}{\beta - i\lambda} + \frac{1}{\beta + i\lambda} \\
&= \frac{2\beta}{\beta^2 + \lambda^2}
\end{aligned}$$

For large β the spektral density does not depend much on the frequencies λ, and so for small λ it can be considered as constant and modelled by a white noise. In fact, for $\beta \to \infty$ the correlation function $k(\tau) = e^{-\beta|\tau|} \to \delta(\tau)$ (i.e. the process becomes δ-correlated).

Question 10

What is the correlation time of a stationary process with auto-correlation $k(\tau) = \sigma^2 e^{-\beta|\tau|}$ (i.e. Gaussian exponentially correlated process)? For which time intervals can we model such a process by a standard white noise?

Answer: *The correlation time is the following integral of the correlation function:*

$$\tau_{\rm cor} = \frac{1}{\sigma^2}\int_0^\infty |k(\tau)|\, d\tau = \int_0^\infty e^{-\beta|\tau|}\, d\tau = \frac{1}{\beta}$$

The correlation time tends to zero $\tau_{\rm cor} \to 0$ as $\beta \to \infty$. If time intervals between events in a system are significantly larger than $\tau_{\rm cor}$, then the process can be modelled by a Gaussian white noise.

Question 11

Which two properties completely characterise a stationary Gaussian stochastic process?

Answer: *The expected (mean) value $\mu = \mathbb{E}\{x(t)\}$ and the auto-correlation function $k(\tau) = \mathbb{E}\{x(t)x(t+\tau)\} - \mu^2$.*