Problems 3: Stochastic processes and nowhere differentiable functions

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Question 1

Which of the functions below are infinitely many times differentiable (i.e. analytic)?

$$f(x) = x\,, \quad f(x) = \frac{x^2}{2}\,, \quad f(x) = e^x\,, \quad f(x) = \ln x\,, \quad f(x) = \cos x$$

Answer: *All of these functions. First three derivatives:*

$$f'(x) = 1\,, \quad f'(x) = x\,, \quad f'(x) = e^x\,, \quad f'(x) = \frac{1}{x}\,, \quad f'(x) = -\sin x$$
$$f''(x) = 0\,, \quad f''(x) = 1\,, \quad f''(x) = e^x\,, \quad f''(x) = -\frac{1}{x^2}\,, \quad f''(x) = -\cos x$$
$$f'''(x) = 0\,, \quad f'''(x) = 0\,, \quad f'''(x) = e^x\,, \quad f'''(x) = \frac{2}{x^3}\,, \quad f'''(x) = \sin x$$

Question 2

For each function above, write its Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Expand $f(x) = \ln x$ at $x_0 = 1$, and all other functions at $x_0 = 0$ (this is called the Maclaurin series).

Answer:

$$
\begin{aligned}
f(x) &= 0 + 1(x-0) + \frac{0^2}{2}(x-0)^2 = x \\
f(x) &= \frac{0^2}{2} + 0(x-0) + \frac{1}{2}(x-0)^2 + 0 + \cdots = \frac{x^2}{2} \\
f(x) &= e^0 + e^0(x-0) + \frac{e^0}{2}(x-0)^2 + \frac{e^0}{6}(x-0)^3 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
f(x) &= \ln 1 + \frac{1}{1}(x-1) - \frac{1}{2\cdot 1^2}(x-1)^2 + \frac{2}{6\cdot 1^3}(x-1)^3 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-1)^n \\
f(x) &= \cos(0) - \sin(0)(x-0) - \frac{\cos(0)}{2!}(x-0)^2 + \frac{\sin(0)}{3!}(x-0)^3 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}
\end{aligned}
$$

Question 3

Why does the existence of a derivative $f'(x)$ at $x = x_0$ requires $f(x)$ to be continuous at $x = x_0$? Hint: use the 'limits' definition of continuity ($\lim_{x\to x_0} f(x) = f(x_0)$).

Answer: *The derivative is defined as a limit of the quotient* $\Delta f(x)/\Delta x = (f(x+\Delta x) - f(x))/\Delta x$, *and this requires the limit* $\lim_{\Delta x\to 0} f(x+\Delta x)$ *to exist and to be equal to* $f(x)$.