Problems 8:
The Black-Scholes theory

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Question 1

The following is the Black-Scholes equation describing how the value $V(S,t)$ of an option depends on the underlying stock price S and time t, when stock is paying continuous dividend at rate ρ:

$$\frac{\partial V}{\partial t} = rV - (r-\rho)S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

Which part of this equation is usually denoted by the Greek letter Δ? What does it represent? How is it used for Δ-hedging? If V is a call option, what is the value of Δ, if S is significantly below the strike price?

Question 2

Consider the Black-Scholes equation in the previous question. Which part of this equation is usually denoted by the Greek letter Γ? What does it represent? How is it used for Γ-hedging? What is the value of Γ, if S is far away from the strike price?

Question 3

The following equations are the Black-Scholes prices of call and put options with stock S paying dividends at constant continuous rate ρ:

$$\begin{aligned} C(S,t) &= e^{-\rho(T-t)}SN(d_1) - e^{-r(T-t)}KN(d_2) \\ P(S,t) &= e^{-r(T-t)}KN(-d_2) - e^{-\rho(T-t)}SN(-d_1) \end{aligned}$$

where

$$d_1 = \frac{\ln(S/K) + (r - \rho + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\,, \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

Differentiate over S to derive the equations for Δ and Γ. Hint: use the fact that the CDF of the standard normal distribution ($\mu = 0$, $\sigma^2 = 1$) is $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-s^2/2}\,ds$, and its derivative is $dN(x)/dx = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

Question 4

Check that the Black-Scholes prices for call and put satisfy the call-put parity: $C - P = e^{-\rho(T-t)}S - e^{-r(T-t)K}$. Hint: use the fact that $N(x) = 1 - N(-x)$ for the CDF of normal distribution.