Problems 4:
Gaussian white noise and Wiener process

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Question 1

Compute characteristic function $\theta(u)$ of the uniform density function $p(x) = \frac{1}{b-a}$ if $x \in [a, b]$, and $p(x) = 0$ otherwise. Hint: use the inverse Fourier transform

$$\theta(u) = \mathcal{F}[f(x)](u) := \int_{-\infty}^{\infty} e^{iux}\, p(x)\, dx$$

Question 2

The characteristic function of Gaussian density $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is

$$\theta(u) = \int_{-\infty}^{\infty} e^{iux} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = e^{iu\mu - \frac{\sigma^2}{2}u^2}$$

Compute all derivatives of the cumulant generating function $\Gamma(u) = \ln\theta(u)$.

Question 3

Use the property $f(0) = \int f(x)\delta(x)\, dx$ of the Dirac δ-function to show that its Fourier transform is a constant function $\mathcal{F}[\delta(x)](y) = 1$.

Question 4

The Dirac δ-function can be approximated by a Gaussian density function with small variance:

$$\delta(x) = \lim_{\sigma\to 0}\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\right)$$

Use this property to prove that Fourier transform of $\delta(x)$ is a constant function.

Question 5

Prove that a δ-correlated process has constant spectral density, and therefore an unbounded (i.e. infinite) variance σ^2. Hint: use the definition of the spektral density as the Fourier transform of the correlation function (in this case $k(\tau) = K\delta(\tau)$, where K is a constant), and the fact that $\sigma^2 = k(0)$.

Question 6

Consider the following probability density function for the values x_1 and x_2 of a stochastic process at two moments in time:

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-R_{12}^2}} e^{-\frac{1}{2(1-R_{12}^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} + 2R_{12}\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]}$$

Prove that x_1 and x_2 are independent if and only if they have zero correlation $R_{12} = 0$.

Question 7

Give the definitions of a Gaussian white noise, standard white noise and the Wiener process.

Question 8

Prove that the Wiener process $w(t)$ is nowhere differentiable in probability (i.e. the probability that time derivative of $w(t)$ exists for some t is zero). Hint: Use the definition of a derivative as the limit of the quotient $\Delta w/\Delta t$ for $\Delta t \to 0$, and the fact that the variance $\mathbb{E}\{(\Delta w)^2\}$ of the increments $\Delta w = w(t + \Delta t) - w(t)$ of the Wiener process $w(t)$ is Δt.

Question 9

What is the spektral density of a stationary process with auto-correlation $k(\tau) = \sigma^2 e^{-\beta|\tau|}$ with $\sigma^2 = 1$ (i.e. Gaussian exponentially correlated process)? For which values of β can we model such a process by a standard white noise?

Question 10

What is the correlation time of a stationary process with auto-correlation $k(\tau) = \sigma^2 e^{-\beta|\tau|}$ (i.e. Gaussian exponentially correlated process)? For which time intervals can we model such a process by a standard white noise?

Question 11

Which two properties completely characterise a stationary Gaussian stochastic process?