Lecture 8: The Black-Scholes theory

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1 Geometric Brownian motion

Geometric Brownian motion

• We are interested in evolution of returns $\Delta S(t)/S(t)$ from stock $S$.

• Notice that

$$y(t) = \ln S(t) \iff dy(t) = \frac{dS(t)}{S(t)}$$

• Stock $S(t)$ follows geometric Brownian motion (GMB) if $y(t) = \ln S(t)$ is described by the SDE:

$$dy(t) = \frac{dS(t)}{S(t)} = \mu dt + \sigma dw(t)$$

• Or equivalently

$$dS(t) = \mu S(t) dt + \sigma S(t) dw(t) = S(t)[\mu dt + \sigma dw(t)]$$
Dynamics of stock

- Apply Ito’s lemma to $F(S) = \ln S$:
  \[
  d\ln S(t) = \left[ \frac{f(S,t)}{S(t)} - \frac{1}{2} \frac{g^2(S,t)}{S^2(t)} \right] dt + \frac{g(S,t)}{S(t)} dw(t)
  \]
  with $f(S,t) = \mu S(t)$ and $g(S,t) = \sigma S(t)$ this reduces to:
  \[
  d\ln S(t) = \left[ \mu - \frac{1}{2} \sigma^2 \right] dt + \sigma dw(t)
  \]
- Thus, for $S(t) = e^{\mu (t)}$
  \[
  S(t) = e^{\left[ \mu - \frac{1}{2} \sigma^2 \right] t + \sigma w(t)} S(0)
  \]

Question 1. How does the value $V$ of a derivative of stock $S$ evolve in time?

2 The Black-Scholes pricing

Outline of derivation from GBM

- Recall that assuming no arbitrage the option price should be the discounted expected value:
  \[
  V(t) = e^{-r(T-t)} \mathbb{E}_P\{V(T) \mid S(t)\}
  \]
  where $V(T) = \max[0, S(T) - K]$ for call ($\max[0, K - S(T)]$ for put).
- Using the GBM model for $S(T)$ we have
  \[
  S(T) = e^{\left[ \mu - \frac{1}{2} \sigma^2 \right] (T-t) + \sigma w(T) - w(t)} S(t)
  \]
  where $\ln(S(T)/S(t))$ has normal distribution $N[(\mu - \sigma^2/2)(T-t), \sigma^2(T-t)]$.
- The expectation $\mathbb{E}_P\{V \mid S(t)\} = \int V(T) dP(S(T) \mid S(t))$ should be taken under the risk-neutral probability, in which case $\ln(S(T)/S(t))$ has distribution $N[(r - \sigma^2/2)(T-t), \sigma^2(T-t)]$.
- The expectation is computed by integral for $S(T)$ such that $V(t) \geq 0$. 

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The Black-Scholes pricing

\( \sigma \) — volatility of stock (i.e. st.dev. \( \sigma \) of \( \ln[S(t + 1)/S(t)] \)).

\( S(t) \) — spot price of stock

\( K \) — strike price

\( r \) — risk-free rate

\( T - t \) — time to expiration

The Black-Scholes formula for European call and put (no dividends)

Assuming \( S \) pays no dividends:

\[
C(t) = S(t)N(d_1) - e^{-r(T-t)}KN(d_2)
\]

\[
P(t) = e^{-r(T-t)}KN(-d_2) - S(t)N(-d_1)
\]

where \( N(x) \) is CDF of normal distribution with zero mean and unit variance \( (N[0, 1]) \) and

\[
d_1 = \frac{\ln(S(t)/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}, \quad d_2 = d_1 - \sigma \sqrt{T - t}
\]

3 The Black-Scholes equation

Black-Scholes preliminaries

- Recall that in a replicating portfolio you ‘replicate’ the value \( V \) of the derivative by investing \( B \) in a riskless bond plus buying or selling \( \Delta \) units of the underlying stock \( S \):

\[
V = B + \Delta S
\]

- A portfolio consisting of options and \( \Delta \) units of stock has the value

\[
\Pi = V - \Delta S
\]

- The idea is to eliminate the risk from this portfolio.

- To know how the value of this portfolio changes with time, we need to know how \( V \) changes with time.

- The value \( V \) of a derivative depends on the underlying stock \( S \) and time \( t \), and therefore we can think of \( V \) as a function \( V(S, t) \).

- Assuming \( V \) is differentiable both in time and stock \( S \), we can apply Ito’s lemma to write \( dV(S, t) \) (we already know how \( S \) evolves).

- Then we eliminate risk by setting \( \Delta = V' = \frac{\partial V}{\partial S} \).
Derivation of the Black-Scholes equation

1. Apply Ito’s lemma to \( V(S,t) \) with \( dS = \mu S dt + \sigma S dw \):

\[
dV(S,t) = \left[ \dot{V} + V' S \mu + \frac{1}{2} V'' S^2 \sigma^2 \right] dt + V' S \sigma dw
\]

2. Substitute \( dV \) and \( dS \) into \( d\Pi = dV - \Delta dS \):

\[
d\Pi = \left[ \dot{V} + V' S \mu + \frac{1}{2} V'' S^2 \sigma^2 - \Delta S \mu \right] dt + [V' - \Delta] S \sigma dw
\]

3. Setting \( \Delta = V' \) eliminates risk, so that \( d\Pi = r \Pi dt = r(V - V'S) dt \)

4. The result is the Black-Scholes equation:

\[
\dot{V} = rV - rV'S - \frac{1}{2} V'' S^2 \sigma^2
\]

The Black-Scholes equation

- Let us rewrite the Black-Scholes equation in more detail:

\[
\frac{\partial V(S,t)}{\partial t} = rV(S,t) - \frac{\partial V(S,t)}{\partial S} S(t) - \frac{1}{2} \frac{\partial^2 V(S,t)}{\partial S^2} S^2(t) \sigma^2
\]

- Or using the Greeks \( \Theta, \Delta \) and \( \Gamma \):

\[
\Theta = rV - r\Delta S(t) - \frac{1}{2} \Gamma S^2(t) \sigma^2
\]

The Greeks

\( \Theta = \frac{\partial V(S,t)}{\partial t} \) — changes of \( V \) over time.
\( \Delta = \frac{\partial V(S,t)}{\partial S} \) — changes of \( V \) with the underlying \( S \) (the slope of \( V(S) \)).
\( \Gamma = \frac{\partial^2 V(S,t)}{\partial S^2} \) — curvature of \( V \) with respect to \( S \).

\( \Delta \) and \( \Gamma \)-hedging

\( \Delta \)-hedging elimination of risk from the option (i.e. from \( dV \)) by buying or selling \( \Delta = \frac{\partial V}{\partial S} \) units of stock. The Black-Scholes equation describes evolution of \( V \) for a strategy with \( \Delta \)-hedging.

\( \Gamma \)-hedging minimization of \( \Gamma \)-risk from an option with high \( \Gamma = \frac{\partial^2 V}{\partial S^2} \). High \( \Gamma \) means that \( \Delta = \frac{\partial V}{\partial S} \) is not optimal for large changes \( \Delta S \) of the underlying stock, leading to the \( \Gamma \)-risk. It can be minimized by adding a third asset to the portfolio (e.g. an option on another stock with its own \( \Gamma \)).
Assumptions of the Black-Scholes theory

The Black-Scholes equation is a powerful tool to price options, but we should not forget the main assumptions, which are:

- Stock $S(t)$ has independent increments $\Delta S(t)$ (i.e. it is a Markov process), which in continuous time means $\Delta S(t)$ is $\delta$-correlated. In reality this is false (otherwise, $\Delta S(t)$ would have infinite variance), but can be assumed on time intervals $\Delta t \gg \tau_{cor}$. For $\Delta t \leq \tau_{cor}$ the theory becomes a very poor representation of reality (e.g. Ito’s lemma cannot be applied).

- Stock $S(t)$ is a stationary process, so that volatility is constant $\sigma(t) = \sigma$. This is false in reality, but can be assumed for relatively short periods $\Delta t$. If time to expiration $T - t \gg \Delta t$, then $\sigma$ cannot be assumed to remain constant (and how does it change then?).

- Risk elimination by $\Delta$-hedging (i.e. $\Delta = \partial V/\partial S$) eliminated stochastic component from $dV$, but it makes the portfolio riskless $d\Pi = r\Pi dt$ only if there is no arbitrage.

- Other assumptions are that $S(t)$ is a continuous-valued and continuous-time process, that risk-free rate $r$ is constant and perhaps a few more.

Reading

- Chapter 7, Sec. 7.6, 7.9 (Elliott & Kopp, 2004).
- Chapter 10 (Roman, 2012)
- Chapter 4, Sec. 4.4, Chapter 8 (Crack, 2014)

References