Lecture 8: The Black-Scholes theory

Dr. Roman V Belavkin

MSO4112

Contents

1 Geometric Brownian motion

Geometric Brownian motion

- We are interested in evolution of returns $\Delta S(t)/S(t)$ from stock S.
- Notice that

$$y(t) = \ln S(t) \qquad \Longleftrightarrow \qquad dy(t) = \frac{dS(t)}{S(t)}$$

- Stock $S(t)$ follows *geometric Brownian motion* (GMB) if $y(t) = \ln S(t)$ is described by the SDE:

$$dy(t) = \frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dw(t)$$

- Or equivalently

$$\begin{aligned} dS(t) &= \mu S(t)\, dt + \sigma S(t)\, dw(t) \\ &= S(t)[\mu\, dt + \sigma\, dw(t)] \end{aligned}$$

Dynamics of stock

- Apply Ito's lemma to $F(S) = \ln S$:

$$d \ln S(t) = \left[\frac{f(S,t)}{S(t)} - \frac{1}{2} \frac{g^2(S,t)}{S^2(t)} \right] dt + \frac{g(S,t)}{S(t)} dw(t)$$

- with $f(S,t) = \mu S(t)$ and $g(S,t) = \sigma S(t)$ this reduces to:

$$d \ln S(t) = \left[\mu - \frac{1}{2}\sigma^2 \right] dt + \sigma \, dw(t)$$

- Thus, for $S(t) = e^{y(t)}$

$$S(t) = e^{\left[\mu - \frac{1}{2}\sigma^2\right]t + \sigma w(t)} S(0)$$

Question 1. *How does the value V of a derivative of stock S evolve in time?*

2 The Black-Scholes pricing

Outline of derivation from GBM

- Recall that assuming no arbitrage the option price should be the discounted expected value:

$$V(t) = e^{-r(T-t)} \mathbb{E}_P\{V(T) \mid S(t)\}$$

  where $V(T)$ is $\max[0, S(T) - K]$ for call ($\max[0, K - S(T)]$ for put).

- Using the GBM model for $S(T)$ we have

$$S(T) = e^{\left[\mu - \frac{1}{2}\sigma^2\right](T-t) + \sigma[w(T) - w(t)]} S(t)$$

  where $\ln(S(T)/S(t))$ has *normal* distribution $N[(\mu - \sigma^2/2)(T-t), \sigma^2(T-t)]$.

- The expectation $\mathbb{E}_P\{V \mid S(t)\} = \int V(T) \, dP(S(T) \mid S(t))$ should be taken under the *risk-neutral* probability, in which case $\ln(S(T)/S(t))$ has distribution $N[(r - \sigma^2/2)(T-t), \sigma^2(T-t)]$.

- The expectation is computed by integral for $S(T)$ such that $V(t) \geq 0$.

The Black-Scholes pricing

σ — volatility of stock (i.e. st.dev. σ of $\ln[S(t+1)/S(t)]$).

$S(t)$ — spot price of stock

K — strike price

r — risk-free rate

$T-t$ — time to expiration

The Black-Scholes formula for European call and put (no dividends)
Assuming S pays no dividends:

$$\begin{aligned} C(t) &= S(t)N(d_1) - e^{-r(T-t)}KN(d_2) \\ P(t) &= e^{-r(T-t)}KN(-d_2) - S(t)N(-d_1) \end{aligned}$$

where $N(x)$ is CDF of normal distribution with zero mean and unit variance ($\mathcal{N}[0,1]$) and

$$d_1 = \frac{\ln(S(t)/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

3 The Black-Scholes equation

Black-Scholes preliminaries

- Recall that in a *replicating portfolio* you 'replicate' the value V of the derivative by investing B in a riskless bond plus buying or selling Δ units of the underlying stock S:

$$V = B + \Delta S$$

- A protfolio consiting of options and Δ units of stock has the value

$$\Pi = V - \Delta S$$

- The idea is to eliminate the risk from this portfolio.
- To know how the value of this portfolio changes with time, we need to know how V changes with time.
- The value V of a derivative depends on the underlying stock S and time t, and therefore we can think of V as a function $V(S,t)$.
- Assuming V is differentiable both in time and stock S, we can apply Ito's lemma to write $dV(S,t)$ (we already know how S evolves).
- Then we *eliminate risk* by setting $\Delta = V' = \frac{\partial V}{\partial S}$.

Derivation of the Black-Scholes equation

1. Apply Ito's lemma to $V(S,t)$ with $dS = \mu S\, dt + \sigma S\, dw$:

$$dV(S,t) = \left[\dot{V} + V' S\, \mu + \frac{1}{2} V'' S^2\, \sigma^2\right] dt + V' S\, \sigma\, dw$$

2. Substitute dV and dS into $d\Pi = dV - \Delta\, dS$:

$$d\Pi = \left[\dot{V} + V' S\, \mu + \frac{1}{2} V'' S^2\, \sigma^2 - \Delta S \mu\right] dt + \underbrace{[V' - \Delta]}_{=0} S\, \sigma\, dw$$

3. Setting $\Delta = V'$ eliminates risk, so that $d\Pi = r\Pi\, dt = r(V - V'S)\, dt$

4. The result is the Black-Scholes equation:

$$\dot{V} = rV - rV'S - \frac{1}{2} V'' S^2 \sigma^2$$

The Black-Scholes equation

- Let us rewrite the Black-Scholes equation in more detail:

$$\underbrace{\frac{\partial V(S,t)}{\partial t}}_{\Theta} = rV(S,t) - r \underbrace{\frac{\partial V(S,t)}{\partial S}}_{\Delta} S(t) - \frac{1}{2} \underbrace{\frac{\partial^2 V(S,t)}{\partial S^2}}_{\Gamma} S^2(t)\, \sigma^2$$

- Or using the Greeks Θ, Δ and Γ:

$$\Theta = rV - r\Delta\, S(t) - \frac{1}{2} \Gamma\, S^2(t)\, \sigma^2$$

The Greeks

$\Theta = \frac{\partial V(S,t)}{\partial t}$ — changes of V over time.

$\Delta = \frac{\partial V(S,t)}{\partial S}$ — changes of V with the underlying S (the slope of $V(S)$).

$\Gamma = \frac{\partial^2 V(S,t)}{\partial S^2}$ — curvature of V with respect to S.

Δ and Γ-hedging

Δ-hedging elimination of risk from the option (i.e. from dV) by buying or selling $\Delta = \frac{\partial V}{\partial S}$ units of stock. The Black-Scholes equation describes evolution of V for a strategy with Δ-hedging.

Γ-hedging minimization of Γ-risk from an option with high $\Gamma = \frac{\partial V^2}{\partial S^2}$. High Γ means that $\Delta = \frac{\partial V}{\partial S}$ is not optimal for large changes ΔS of the underlying stock, leading to the Γ-risk. It can be minimized by adding a third asset to the portfolio (e.g. an option on another stock with its own Γ).

Assumptions of the Black-Scholes theory

The Black-Scholes equation is a powerful tool to price options, but we should not forget the main assumptions, which are:

- Stock $S(t)$ has independent increments $\Delta S(t)$ (i.e. it is a Markov process), which in continuous time means $\Delta S(t)$ is *δ-correlated*. In reality this is false (otherwise, $\Delta S(t)$ would have infinite variance), but can be assumed on time intervals $\Delta t \gg \tau_{\rm cor}$. For $\Delta t \leq \tau_{\rm cor}$ the theory becomes a very poor representation of reality (e.g. Ito's lemma cannot be applied).

- Stock $S(t)$ is a *stationary* process, so that volatility is constant $\sigma(t) = \sigma$. This is false in reality, but can be assumed for relatively short periods Δt. If time to expiration $T - t \gg \Delta t$, then σ cannot be assumed to remain constant (and how does it change then?).

- Risk elimination by Δ-hedging (i.e. $\Delta = \partial V/\partial S$) eliminated stochastic component from dV, but it makes the portfolio riskless $d\Pi = r\Pi\, dt$ only if there is *no arbitrage*.

- Other assumptions are that $S(t)$ is a continuous-valued and continuous-time process, that risk-free rate r is constant and perhaps a few more.

Reading

- Chapter 7, Sec. 7.6, 7.9 (Elliott & Kopp, 2004).

- Chapter 10 (Roman, 2012)

- Chapter 4, Sec. 4.4, Chapter 8 (Crack, 2014)

References

Crack, T. F. (2014). *Basic Black-Scholes: Option pricing and trading* (3rd ed.). Timothy Crack.

Elliott, R. J., & Kopp, P. E. (2004). *Mathematics of financial markets* (2nd ed.). Springer.

Roman, S. (2012). *Introduction to the mathematics of finance: Arbitrage and option pricing*. Springer.