Lecture 7: Ito differentiation rule

Dr. Roman V Belavkin

MSO4112

Contents

1 Classical differential df and the rule $dt^2 = 0$

Classical differential df

- Let $F(t)$ be a function of time $t \in [0, T]$.
- The increment of the value of f during Δt is

$$\Delta F(t) = F(t + \Delta t) - F(t)$$

- Recall that the derivative $dF(t)/dt$ is $\lim_{\Delta t \to 0} \Delta F(t)/\Delta t$
- The differential $dF(t)$ can be thought of as the increment $\Delta F(t)$ during infinitessimal dt:

$$dF(t) = F(t + dt) - F(t)$$

- We can show that this corresponds to the formal rule $dt^2 = 0$.

Newton's rule $dt^2 = 0$

- Assume that $F(t + \Delta t)$ can be computed as Taylor series at time t:

$$F(t + \Delta t) = F(t) + \frac{dF(t)}{dt}\Delta t + \frac{1}{2}\frac{d^2F(t)}{dt^2}\Delta t^2 + \frac{1}{6}\frac{d^3F(t)}{dt^3}\Delta t^3 + \cdots$$

- This gives the following formula for the increment $\Delta F(t)$:

$$F(t+\Delta t) - F(t) = \frac{dF(t)}{dt}\Delta t + \frac{1}{2}\frac{d^2F(t)}{dt^2}\Delta t^2 + \frac{1}{6}\frac{d^3F(t)}{dt^3}\Delta t^3 + \cdots$$

- Now consider the limit $\Delta t \to dt$:

$$F(t+dt) - F(t) = \frac{dF(t)}{dt}\,dt + \underbrace{\frac{1}{2}\frac{d^2F(t)}{dt^2}\,dt^2 + \frac{1}{6}\frac{d^3F(t)}{dt^3}\,dt^3 + \cdots}_{=0}$$

- Observe that the rule $dt^2 = 0$ above corresponds to the formula for the differential below:

$$F(t+dt) - F(t) = dF(t)$$

Differential $dF(x,t)$

- Let $F(x,t)$ be a function of t and signal $x(t)$, and denote by $\dot{F}$, $\ddot{F}$, $\dddot{F}$, ... time derivatives, and by F', F'', F''', ... derivatives over x.
- Assume that $F(x(t+dt), t+\Delta t)$ has Taylor expansion at (x,t):

$$\begin{aligned} &F(x(t+dt), t+dt) = \\ &\qquad F(x,t) + \dot{F}(x,t)\,dt + \underbrace{\frac{1}{2}\ddot{F}(x,t)\,dt^2 + \cdots}_{dt^2=0} \\ &\qquad +F'(x,t)\,dx + \frac{1}{2}F''(x,t)\,dx^2 + \underbrace{\frac{1}{2}\dot{F}'(x,t)\,dt\,dx + \cdots}_{dt\,dx=0} \end{aligned}$$

- Observe that rules $dt^2 = 0$ and $dt\,dx = 0$ lead to the following formula for the differential $dF(x,t) = F(x(t+dt), t+dt) - F(x,t)$:

$$dF(x,t) = \dot{F}(x,t)\,dt + F'(x,t)\,dx + \frac{1}{2}F''(x,t)\,dx^2$$

- Can we assume also $dx^2 = 0$?

2 Stochastic differential $dx^2 \neq 0$ and $dw^2 = dt$

Stochastic differential $dx^2 \neq 0$

- If signal $x(t)$ has time derivative $\dot{x}(t) = dx(t)/dt$, then $dx(t) = \dot{x}(t)\,dt$ and

$$dx^2(t) = [\dot{x}(t)\,dt]^2 = \dot{x}^2(t)\,dt^2 = 0$$

- If, on the other hand, $x(t)$ is nowhere differentiable (e.g. stochastic), then generally $dx^2(t) \neq 0$.
- For example, if $x(t)$ is described by an SDE:

$$dx(t) = f(x,t)\, dt + g(x,t)\, dw$$

- then for dx^2 we have

$$\begin{aligned} dx^2(t) &= [f(x,t)\, dt + g(x,t)\, dw]^2 \\ &= f^2(x,t)\, \underbrace{dt^2}_{=0} + 2f(x,t)\, g(x,t)\, \underbrace{dt\, dw}_{=0} + g^2(x,t)\, \underbrace{dw^2}_{=dt} \\ &= g^2(x,t)\, dt \end{aligned}$$

- Where we used the Levy's substitution rule $dw^2 = dt$.

The Levy rule $dw^2 = dt$

Theorem 1 (Levy)**.** *The following substitutions are valid in the difference schemes*

$$\begin{aligned} \Delta w^2(t) &\mapsto \mathbb{E}\{\Delta w^2(t)\} = \Delta t \\ \Delta x^2(t) &\mapsto \mathbb{E}\{\Delta x^2(t)\} = g^2(x(t),t)\Delta t + O(\Delta t) \end{aligned}$$

and differentials

$$\begin{aligned} dw^2(t) &\mapsto \mathbb{E}\{dw^2(t)\} = dt \\ dx^2(t) &\mapsto \mathbb{E}\{dx^2(t)\} = g^2(x(t),t)\, dt \end{aligned}$$

- The proof is based on the property $\mathbb{E}\{w^2(t)\} = t$ of the Wiener process.
- Notice the use of the expected values $\mathbb{E}$, which means that, strictly speaking, these substitutions should be understood in the 'almost sure' sense.

3 Ito' lemma

Ito's lemma

- Because $dx^2(t) \neq 0$ in general, we have to use the following formula for the differential $dF(x,t)$:

$$dF(x,t) = \dot{F}\, dt + F'\, dx(t) + \frac{1}{2} F''\, dx^2(t)$$

- We also derived that for $x(t)$ satisfying SDE $dx(t) = f(x,t)\, dt + g(x,t)\, dw(t)$:

$$dx^2(t) = g^2(x,t)\, dt$$

- Substituting $dx(t)$ and $dx^2(t)$ into $dF(x,t)$ we obtain:

Lemma 2 (Ito)**.**

$$\begin{aligned} dF(x,t) &= \left[\dot{F} + \frac{1}{2}F''\, g^2(x,t)\right] dt + F'\, dx(t) \\ &= \left[\dot{F} + F'\, f(x,t) + \frac{1}{2}F''\, g^2(x,t)\right] dt + F'\, g(x,t)\, dw(t) \end{aligned}$$

Generalised differentiation rule

- If we use general difference schemes $d_\lambda x$ such that $x(t)$ satisfies general SDE
$$d_\lambda x(t) = f(x,t)\, dt + g(x,t)\, d_\lambda w(t)$$
- then the differentiation rule is:
$$d_\lambda F(x,t) = \left[\dot{F} + \left(\frac{1}{2} - \lambda\right) F''\, g^2(x,t)\right] dt + F'\, d_\lambda x(t)$$
- In the Stratonovich case $\lambda = \frac{1}{2}$
$$d_{\frac{1}{2}} F(x,t) = \dot{F}\, dt + F'\, d_{\frac{1}{2}} x(t)$$

Example

- Find SDE for $y(t) = \ln x(t)$, where $dx(t) = f(x,t)\, dt + g(x,t)\, dw$.
- For $F(x,t) = \ln x(t)$ we have:
$$\dot{F} = 0\,, \quad F' = \frac{1}{x(t)}\,, \quad F'' = -\frac{1}{x^2(t)}$$
- Applying Ito's lemma
$$d\ln x(t) = \left[\frac{f(x,t)}{x(t)} - \frac{1}{2}\frac{g^2(x,t)}{x^2(t)}\right] dt + \frac{g(x,t)}{x(t)}\, dw(t)$$
- Let $f(x,t) = ax(t)$ and $g(x,t) = bx(t)$. Then
$$d\ln x(t) = \left[a - \frac{1}{2}b^2\right] dt + b\, dw(t)$$
- Therefore $x(t) = e^{(a-b^2/2)t + bw(t)}x(0)$.

Reading

- Chapter 6, Sec. 6.4 (Elliott & Kopp, 2004).

References

Elliott, R. J., & Kopp, P. E. (2004). *Mathematics of financial markets* (2nd ed.). Springer.