Lecture 6: Stochastic differential equations and integrals

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1 Stochastic differential equations

Stochastic differential equations

Definition 1 (Stochastic differential equations)**.** Equations of the form

$$dx(t) = f(x,t)\,dt + g(x,t)\,dw(t)$$

where $w(t)$ is the Wiener process, and $f(x,t)$, $g(x,t)$ are some functions.

- Compare an SDE to an ordinary differential equation:

$$\frac{dx(t)}{dt} = f(x,t) \quad \text{or} \quad dx(t) = f(x,t)\,dt$$

- Why cannot we write an SDE using time derivatives dx/dt, dw/dt?
- Hint: recall that the Wiener process has the property:

$$dw(t) \sim \sqrt{dt}$$

  so that dw/dt is proportional to $1/\sqrt{dt} \to \infty$.

Question 1. *Is a stochastic process $x(t)$, described by an SDE with $g \neq 0$, differentiable at any t?*

2 Solutions to SDE

Solutions to an SDE

- The existence and uniqueness of solution to an SDE requires some conditions on funcitons f and g.
- If function $f(x,t)$ is Lipshitz continnuous: $|f(x_1,t)-f(x_2,t)| < A|x_1-x_2|$
- and if $g(x,t)$ is bounded and smooth
- then an SDE has unique solution

  $$x(t) = x(0) + \int_0^t f(x,s)\,ds + \int_0^t g(x,s)\,dw(s)$$

  where $\int_0^t g(x,s)\,dw(s)$ is a *stochastic* integral (the role of a measure over which we integrate is played by the Wiener process, which, remember, is nowhere differentiable).

3 Stochastic integrals

Riamann vs Stochastic integral

- Let us consider an integral over a differentiable function $u(t)$.
- Then the Riemann integral is defined by the limit:

  $$\int_0^t g(x,s)\,du(s) = \lim_{\Delta t \to 0} \sum_{i=1}^{M-1} g(x(t_i),t_i)\,\Delta u(t_i)$$

  where $\Delta u(t_i) = u(t_i + \Delta t) - u(t_i)$
- As $\Delta t \to 0$, the difference $\Delta u(t_i) = u(t_i + \Delta t) - u(t_i)$ tends the unique limit $du(t_i) = \dot{u}(t_i)\,dt$.
- This does not depend on whether we take *forward* or *backward* differences:

  $$\lim_{\Delta t \to 0} u(t_i + \Delta t) - u(t_i) = \lim_{\Delta t \to 0} u(t_i) - u(t_i - \Delta t)$$

- This is not the case if, instead of differentiable $u(t)$, we integrate over nowhere differentiable $w(t)$.

Difference schemes

We can consider the following difference schemes, each leading to a different result in the limit $\Delta t \to 0$:

- Ito (forward):
$$\Delta_0 x(t) = x(t_{i+1}) - x(t_i)$$
- Backward:
$$\Delta_1 x(t) = x(t_i) - x(t_{i-1})$$
- General (mixture of forward and backward):
$$\Delta_\lambda x(t) = (1-\lambda)[x(t_{i+1}) - x(t_i)] + \lambda[x(t_i) - x(t_{i-1})]$$
- Stratonovich (or balanced scheme with $\lambda = 1/2$):
$$\Delta_s x(t) = \frac{x(t_{i+1}) - x(t_{i-1})}{2}$$

Ito and Stratonovich Integrals

Definition 2 (General stochastic integral)**.** over stochastic process $x(t)$ is the limit of the integral sum $\sum G(x,t)\,\Delta_\lambda x(t)$ with the general difference scheme $\Delta_\lambda x(t)$:
$$I_\lambda[G(x), dx] = \int_0^t G(x,s)\,d_\lambda x(s)$$

- Ito integral uses the forward scheme ($\lambda = 0$).
- Statonovich integral uses symmetrised scheme ($\lambda = \frac{1}{2}$).
- For $dx(t) = f(x,t)\,dt + g(x,t)\,dw$, they are related as follows:
$$\int_0^t G(x,s)\,d_{\frac{1}{2}}x(s) = \int_0^t G(x,s)\,d_0 x(s) + \frac{1}{2}\int_0^t \frac{\partial G(x,s)}{\partial x}\,g^2(x,t)\,dt$$
- Notice what happens if $dx(t) = f(x,t)\,dt$ (i.e. if $g = 0$).

Reading

- Chapter 6, Sec. 6.3, 6.5 (Elliott & Kopp, 2004).

References

Elliott, R. J., & Kopp, P. E. (2004). *Mathematics of financial markets* (2nd ed.). Springer.