Lecture 5: Continuous Markov process and the diffusion equation

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Contents

1 Markov property

Markov property

- Markov propety (no memory or aftereffect):

$$p(x_{t+1} \mid x_t, \dots, x_1) = p(x_{t+1} \mid x_t)$$

- The conditional probability density $p(x_{t+1} \mid x_t)$ is called *transition function* (or transition probability).

- The joint probability density of a Markov process is computed using a product of transition probabilities:

$$\begin{aligned} & p(x_{t+1}, x_t, \dots, x_1) \\ &= \underbrace{p(x_{t+1} \mid x_t, \dots, x_1)}_{p(x_{t+1} \mid x_t)} \underbrace{p(x_t \mid x_{t-1}, \dots, x_1)}_{p(x_t \mid x_{t-1})} \cdots \underbrace{p(x_2 \mid x_1)}_{p(x_2 \mid x_1)} p(x_1) \end{aligned}$$

Example 1 (Independent increments). Any process $\{x(t)\}_{t\in[0,T]}$ with independent increments $\Delta x(t) = x(t+1) - x(t)$ is Markov. Thus, the Wiener process is Markov (it is a 'sum' of white noise, which is a δ-correlated process).

2 Kolmogorov-Chapman equation

Kolmogorov-Chapman equation

- Recall the law of total probability:

$$p(x) = \int p(x, y)\, dy\,, \quad p(x, y) = \int p(x, y, z)\, dz\,, \ldots$$

- Write $p(x_{t+2}, x_{t+1}, x_t) = p(x_{t+2} \mid x_{t+1}) p(x_{t+1} \mid x_t) p(x_t)$ with $p(x_t) = \delta$, and integrate over x_{t+1}:

$$p(x_{t+2} \mid x_t) = \int p(x_{t+2} \mid x_{t+1}) p(x_{t+1} \mid x_t)\, dx_{t+1}$$

Question 1. *How do $p(x_{t+\tau} \mid x_t)$ or $p_\tau(x_\tau) = \int p(x_{t+\tau} \mid x_t) p(x_t)\, dx_t$ depend on τ? Can we find the time derivative:*

$$\dot{p}(x) = \lim_{\tau \to 0} \frac{p_\tau(x) - p(x)}{\tau}$$

3 Kinetic equation

Kinetic equation

- Consider a transformation of $p(x)$ into $p_\tau(x_\tau)$:

$$\begin{aligned} p_\tau(x_\tau) &= \int p(x_\tau \mid x) p(x)\, dx \\ &= \frac{1}{2\pi} \int \int e^{-iu(x_\tau - x)} \Theta(u, x)\, du\, p(x)\, dx \\ &= \frac{1}{2\pi} \int \int e^{-iu(x_\tau - x)} \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} m_n(x))\, du\, p(x)\, dx \\ &= p(x_\tau) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x_\tau} \right)^n [m_n(x_\tau) p(x_\tau)] \end{aligned}$$

- Taking $\lim_{\tau \to 0} \frac{p_\tau(x_\tau) - p(x_\tau)}{\tau}$:

$$\dot{p}(x) = \sum_{s=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x_\tau} \right)^n [K_n(x) p(x)]\,, \quad K_n(x) = \lim_{\tau \to 0} \frac{m_n(x)}{\tau}$$

4 Fokker-Planck equation

Fokker-Planck equation

Definition 2 (Continuous Markov process)**.**

- A Markov process, for which $K_n = 0$ for all $n > 2$.
- In this case, the kinetic equation is

$$\dot{p}(x) = -\frac{\partial}{\partial x}[K_1(x)p(x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[K_2(x)p(x)]$$

- Coefficient $K_1(x)$ is called *drift*.
- Coefficient $K_2(x)$ is called *diffusion*.

Example 3 (Diffusion equation). If the drift $K_1 = 0$ and diffusion $K_2 = 1$, then

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}$$

This corresponds to Wiener process $\{x(t)\}_{t\in(0,T)}$.

Solution of the Fokker-Planck Equation

- For stationary $K_1(x)$ and $K_2(x)$, the solution is a stationary process:

$$p(x) = \frac{C}{K_2(x)} \exp\left\{2\int_{x_1}^{x} \frac{K_1(y)}{K_2(y)}\,dy\right\}$$

- If $K_1(x) = -\alpha x$ and $K_2(x) = g^2$, then the solution is Gaussian process $N_x[0, g^2/2\alpha]$ (zero mean and variance $g^2/2\alpha$):

$$p(x) = \sqrt{\frac{\alpha}{\pi g^2}} e^{-\frac{\alpha x^2}{g^2}}$$

Reading

- Chapter 7 (Crack, 2014)
- Chapter 4 (Stratonovich, 2014)

References

Crack, T. F. (2014). *Basic Black-Scholes: Option pricing and trading* (3rd ed.). Timothy Crack.

Stratonovich, R. L. (2014). *Topics in the theory of random noise* (Vol. 1). Martino Fine Books.