Lecture 4: Gaussian white noise and Wiener process

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Contents

1 Gaussian process

Gaussian stochastic process

- If for arbitrary partition $\{t_1, \dots, t_n\} \subset (0, T)$, the density of $\{x_1, \dots, x_n\}$ is Gaussian:

$$p(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \|k_{ij}\|}} e^{-\frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n a_{ij}(x_i - \bar{x}_i)(x_j - \bar{x}_j)}$$

  where $\bar{x}_i = \mathbb{E}\{x_i\}$ are the *mean* values and

$$k_{ij} = \mathbb{E}\{(x_i - \bar{x}_i)(x_j - \bar{x}_j)\} = \mathbb{E}\{x_i x_j\} - \bar{x}_i \bar{x}_j$$

  are the *covariances*. They completely define a Gaussian process.

- The matrix $\|a_{ij}\|$ is the inverse $\|k_{ij}\|^{-1}$ of the covariance matrix.

Example 1.

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - R_{12}^2}} e^{-\frac{1}{2(1-R_{12}^2)}\left[\frac{(x_1-\bar{x}_1)^2}{\sigma_1^2} + \frac{(x_2-\bar{x}_2)^2}{\sigma_2^2} + 2R_{12}\frac{(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{\sigma_1\sigma_2}\right]}$$

where $R_{12} = \frac{k_{12}}{\sigma_1\sigma_2}$ is the correlation coefficient.

2 White noise

White noise

- $\{x(t)\}_{t\in(0,T)}$ is called Gaussian δ-*correlated* process, if it has the following correlation function:

$$k(\tau) = N\delta(\tau)\,, \qquad N = \text{const}$$

- It has a constant power spectrum:

$$S[x,\lambda] = N\int_{-\infty}^{\infty} \delta(\tau)e^{-i\lambda\tau}\,d\tau = N$$

- $\{x(t)\}_{t\in(0,T)}$ is called *standrad white noise* if $N = 1$, si that

$$\mathbb{E}\{x(t)\} = 0\,, \qquad k(\tau) = \mathbb{E}\{x(t)x(t+\tau)\} = \delta(\tau)$$

  and therefore $S[x,\lambda] = 1$ ($N = 1$).

Exponentially correlated process

- Stationary Gaussian process with exponential correlation function

$$k(\tau) = \sigma^2 e^{-\alpha|\tau|}\,, \qquad \alpha = \text{const}$$

- Its power spectrum is

$$\begin{aligned} S[x,\lambda] &= \sigma^2\left[\int_0^{\infty} e^{-(\alpha+i\lambda)\tau}\,d\tau + \int_{-\infty}^{0} e^{(\alpha-i\lambda)\tau}\,d\tau\right] \\ &= \sigma^2\left[\frac{1}{\alpha+i\lambda} + \frac{1}{\alpha-i\lambda}\right] = \frac{2\sigma^2\alpha}{\alpha^2+\lambda^2} \end{aligned}$$

- We can write $S[x,\lambda] = \frac{\alpha^2}{\alpha^2+\lambda^2}S[x,0]$, where $S[x,0] = 2\sigma^2/\alpha$.
- The correlation time for this process is

$$\tau_{\text{cor}} = \frac{1}{k(0)}\int_0^{\infty} |k(\tau)|\,d\tau = \alpha^{-1}$$

- As $\alpha \to \infty$, $\sigma^2 = S[x,0]\alpha/2 \to \infty$ and $\tau_{cor} \to 0$ (i.e. the process becomes white noise).

3 Linear transformation of white noise

Linear transformation of white noise

- Input $x(t)$ and output $y(t)$ related by $L_l y(t) = M_m x(t)$
- First order linear stationary system

$$\dot{y}(t) = -\alpha y(t) + g x(t)\,, \quad y(0) = y_0$$

- Solution

$$y(t) = e^{-\alpha t} y(0) + g \int_0^t e^{-\alpha(t-s)} x(s)\, ds$$

- If $x(t)$ is white noise, then $\mathbb{E}\{y(t)\} = e^{-\alpha t} y(0)$ and

$$\begin{aligned} k_y(t_1, t_2) &= g^2 \int_0^{t_1} \int_0^{t_2} e^{-\alpha(t_1 - s)} e^{-\alpha(t_2 - r)} \delta(s - r)\, ds\, dr \\ &= g^2 e^{-\alpha(t_1 + t_2)} \int_0^t e^{\alpha s}\, ds \int_0^{t+\tau} e^{\alpha r} \delta(s - r)\, dr = \sigma^2(t) e^{-\alpha \tau} \end{aligned}$$

  where $\sigma^2(t) = \frac{g^2}{2\alpha}(1 - e^{-2\alpha t})$.

4 Wiener process

Wiener process

- $\{w(t)\}_{t \in (0,T)}$ is the integral of white noise $x(\tau)$ on $\tau \in [0, t]$:

$$w(t) = \int_0^t x(\tau)\, d\tau = \lim_{\Delta t \to 0} \sum_{m=1}^{M-1} x\left(\frac{\tau_m + \tau_{m+1}}{2}\right) \Delta t$$

  where $x(t)$ is standard Gaussian white noise, that is $\mathbb{E}\{x\} = 0$ and $k(\tau) = N\delta(\tau)$, $N = 1$, $S[\lambda] = N = 1$.

- One can show that $\mathbb{E}\{w(t)\} = 0$ and $\sigma^2(t) = t$.
- Based on $\sigma^2(t) = k_y(t_1, t_1)$ with $g = 1$ and taking the limit as $\alpha \to 0$:

$$\lim_{\alpha \to 0} \frac{1}{2\alpha}(1 - e^{-2\alpha t}) = t$$

- This gives properties $\mathbb{E}\{\Delta w^2(t)\} = \Delta t$ and:

$$dw(t) \sim \sqrt{dt}\,, \quad \dot{w}(t) \sim 1/\sqrt{dt}$$

Reading

- Chapter 6, Sec. 6.2, 6.4 (Elliott & Kopp, 2004).
- Chapter 10 (Roman, 2012)

References

Elliott, R. J., & Kopp, P. E. (2004). *Mathematics of financial markets* (2nd ed.). Springer.

Roman, S. (2012). *Introduction to the mathematics of finance: Arbitrage and option pricing*. Springer.