

Lecture 2: Binomial pricing

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1 One step model

Present value of options

- If there is no arbitrage, then the present value of options is

$$V(t) = e^{-r(T-t)}\mathbb{E}\{V(T)\}$$

- The final payoffs for long call and long put are:

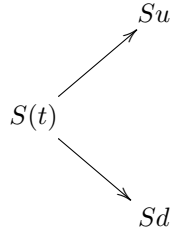
$$V(T) = \max[0, S(T) - K], \quad V(T) = \max[0, K - S(T)]$$

- The expected value $\mathbb{E}\{V(T)\}$ is a sum or an integral, which depend on the prices of stock $S \in [0, +\infty)$ and their probabilities $P(S)$:

$$\begin{aligned}\mathbb{E}\{V(T)\} &= \int_0^\infty \max[0, S(T) - K] dP(S) \\ &= \underbrace{\int_K^\infty S dP(S)}_{\mathbb{E}\{S|S \geq K\}} - K\end{aligned}$$

- Thus, all we need is the *probability distribution* of stock prices $P(S)$ at the expiration time T .

Risk-neutral probabilities



- Assume that $S(t)$ can only change up $S(T) = Su$ ($u > 1$) or down $S(T) = Sd$ ($d < 1$) with probabilities $P(Su) = \lambda$ and $P(Sd) = 1 - \lambda$:
- Then the expected value is

$$\mathbb{E}\{S(T)\} = Su\lambda + Sd(1 - \lambda)$$

- Assuming no arbitrage, the expected stock price at T is:

$$\mathbb{E}\{S(T)\} = e^{r(T-t)}S(t)$$

- The equation $Su\lambda + Sd(1 - \lambda) = e^{r(T-t)}S$ gives

$$\lambda = \frac{e^{r(T-t)} - d}{u - d}$$

One-step binomial pricing

- We can also value an option:

$$V(t) = e^{-r(T-t)}[V_u\lambda + V_d(1 - \lambda)]$$

where

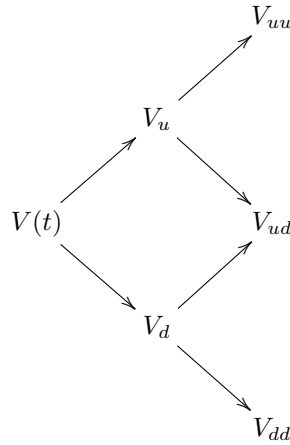
- V_u — final payoff for $S(T) = Su$.
- V_d — final payoff for $S(T) = Sd$.

Example 1. • Suppose $S(t) = \pounds 50$ can change to $Su = \pounds 60$ or $Sd = \pounds 40$. Assuming $r = 0$, what is the value of European call with $K = \pounds 55$?

- Computing $u = 1.2$, $d = .8$, $\lambda = 1/2$, and using $V_u = \max[0, Su - K]$, $V_d = \max[0, Sd - K]$, you should obtain $V(t) = \pounds 2.5$.

2 N -step model

2-step model



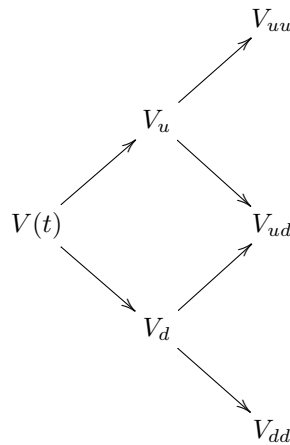
- Divide $T - t$ into $N = 2$ steps $\Delta t = \frac{T-t}{2}$.
- Value each step:

$$\begin{aligned} V(t) &= e^{-r(T-t)/2} [V_u \lambda + V_d (1 - \lambda)] \\ V_u &= e^{-r(T-t)/2} [V_{uu} \lambda + V_{ud} (1 - \lambda)] \\ V_d &= e^{-r(T-t)/2} [V_{du} \lambda + V_{dd} (1 - \lambda)] \end{aligned}$$

- Risk-neutral probability for $N = 2$ is

$$\lambda = \frac{e^{r \frac{(T-t)}{2}} - d}{u - d}$$

N -step model



- Divide $T - t$ into N steps $\Delta t = \frac{T-t}{N}$.
- For N steps:

$$V(t) = e^{-r(T-t)} \underbrace{\sum_{i=0}^N \binom{N}{i} \lambda^i (1-\lambda)^{N-i} V_{u^i d^{N-i}}}_{\mathbb{E}\{V\} \text{ after } N \text{ steps}}$$

- Risk-neutral probability for N steps:

$$\lambda = \frac{e^{r \frac{(T-t)}{N}} - d}{u - d}$$

N -step call option

- Given a specific option, one can simplify the formula.
- For a call with K and stock $S(t)$, the payoff is $V(T) = \max[0, S(T) - K]$.
- Let n be the smallest number of steps such that $S(t)u^n - K \geq 0$.
- Then the value $C(t)$ of call option is

$$C(t) = S(t)F(n, N, \lambda) - e^{-r(T-t)} K F(n, N, \bar{\lambda})$$

where F is computed using binomial distribution:

$$F(n, N, \lambda) = \sum_{i=n}^N \binom{N}{i} \lambda^i (1-\lambda)^{N-i}$$

and $\bar{\lambda} = u e^{-r \frac{(T-t)}{N}} \lambda$.

- Given volatility σ , one usually sets $u = e^{\sigma\sqrt{T-t}}$ and $d = e^{-\sigma\sqrt{T-t}}$.
- In the limit $N \rightarrow \infty$ binomial pricing corresponds to Black-Scholes formula.

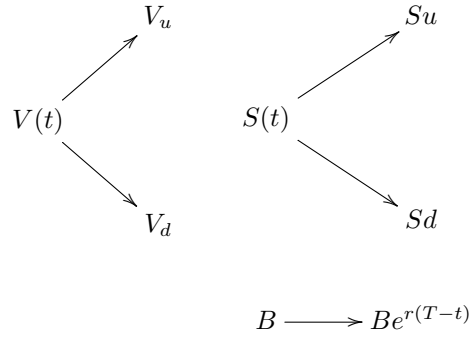
3 Replicating portfolio

Replicating portfolio

- We would like to buy X amount of stock at price $S(t)$ and invest B into a riskless bond to replicate the value V of an option:

$$V(t) = S(t)X + B$$

- At T the values change as follows



Replicating portfolio for one-step binomial

- Using $V_u = S(t)uX + Be^{r(T-t)}$ and $V_d = S(t)dX + Be^{r(T-t)}$ gives

$$X = \frac{V_u - V_d}{S(u - d)}, \quad B = \frac{uV_d - dV_u}{e^{r(T-t)}(u - d)}$$

- Substituting into $V(t) = S(t)X + B$ gives the price of option:

$$V(t) = e^{-r(T-t)}[V_u\lambda + V_d(1 - \lambda)]$$

where $\lambda = \frac{e^{r(T-t)} - d}{u - d}$.

Reading

- Chapter 1, Sec. 1.3–1.5 (Elliott & Kopp, 2004).
- Chapter 5, 6 (Roman, 2012)
- Chapter 6, 8, Sec. 8.3.3 (Crack, 2014)

References

- Crack, T. F. (2014). *Basic Black-Scholes: Option pricing and trading* (3rd ed.). Timothy Crack.
- Elliott, R. J., & Kopp, P. E. (2004). *Mathematics of financial markets* (2nd ed.). Springer.
- Roman, S. (2012). *Introduction to the mathematics of finance: Arbitrage and option pricing*. Springer.