Lecture 2: Binomial pricing

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Contents

1 One step model

Present value of options

- If there is no arbitrage, then the present value of options is

$$V(t) = e^{-r(T-t)}\mathbb{E}\{V(T)\}$$

- The final payoffs for long call and long put are:

$$V(T) = \max[0, S(T) - K]\,, \quad V(T) = \max[0, K - S(T)]$$

- The expected value $\mathbb{E}\{V(T)\}$ is a sum or an integral, which depend on the prices of stock $S \in [0, +\infty)$ and their probabilities $P(S)$:

$$\begin{aligned}
\mathbb{E}\{V(T)\} &= \int_0^\infty \max[0, S(T) - K]\, dP(S) \\
&= \underbrace{\int_K^\infty S\, dP(S)}_{\mathbb{E}\{S|S\geq K\}} - K
\end{aligned}$$

- Thus, all we need is the *probability distribution* of stock prices $P(S)$ at the expiration time T.

Risk-neutral probabilities

$$\begin{tikzcd}[row sep=small] & Su \\ S(t) \arrow[ur] \arrow[dr] & \\ & Sd \end{tikzcd}$$

- Assume that $S(t)$ can only change up $S(T) = Su$ $(u > 1)$ or down $S(T) = Sd$ $(d < 1)$ with probabilities $P(Su) = \lambda$ and $P(Sd) = 1 - \lambda$:
- Then the expected value is

$$\mathbb{E}\{S(T)\} = Su\,\lambda + Sd\,(1 - \lambda)$$

- Assuming no arbitrage, the expected stock price at T is:

$$\mathbb{E}\{S(T)\} = e^{r(T-t)}S(t)$$

- The equation $Su\,\lambda + Sd\,(1 - \lambda) = e^{r(T-t)}S$ gives

$$\lambda = \frac{e^{r(T-t)} - d}{u - d}$$

One-step binomial pricing

- We can also value an option:

$$V(t) = e^{-r(T-t)}[V_u\lambda + V_d(1 - \lambda)]$$

  where
- Vu — final payoff for $S(T) = Su$.
- Vd — final payoff for $S(T) = Sd$.

Example 1.
- Suppose $S(t) = £50$ can change to $Su = £60$ or $Sd = £40$. Assuming $r = 0$, what is the value of European call with $K = £55$?
- Computing $u = 1.2$, $d = .8$, $\lambda = 1/2$, and using $V_u = \max[0, Su - K]$, $V_d = \max[0, Sd - K]$, you should obtain $V(t) = £2.5$.

2 N-step model

2-step model

$$\begin{tikzcd}
 & & V_{uu} \\
 & V_u \arrow[ur] \arrow[dr] & \\
V(t) \arrow[ur] \arrow[dr] & & V_{ud} \\
 & V_d \arrow[ur] \arrow[dr] & \\
 & & V_{dd}
\end{tikzcd}$$

- Divide $T-t$ into $N=2$ steps $\Delta t = \frac{T-t}{2}$.
- Value each step:

$$\begin{aligned}
V(t) &= e^{-r(T-t)/2}[V_u\lambda + V_d(1-\lambda)] \\
V_u &= e^{-r(T-t)/2}[V_{uu}\lambda + V_{ud}(1-\lambda)] \\
V_d &= e^{-r(T-t)/2}[V_{du}\lambda + V_{dd}(1-\lambda)]
\end{aligned}$$

- Risk-neutral probability for $N=2$ is

$$\lambda = \frac{e^{r\frac{(T-t)}{2}} - d}{u-d}$$

N-step model

$$\begin{tikzcd}
 & & V_{uu} \\
 & V_u \arrow[ur] \arrow[dr] & \\
V(t) \arrow[ur] \arrow[dr] & & V_{ud} \\
 & V_d \arrow[ur] \arrow[dr] & \\
 & & V_{dd}
\end{tikzcd}$$

- Divide $T-t$ into N steps $\Delta t = \frac{T-t}{N}$.
- For N steps:

$$V(t) = e^{-r(T-t)} \underbrace{\sum_{i=0}^{N} \binom{N}{i} \lambda^i (1-\lambda)^{N-i} V_{u^i d^{N-i}}}_{\mathbb{E}\{V\} \text{ after } N \text{ steps}}$$

- Risk-neutral probability for N steps:

$$\lambda = \frac{e^{r\frac{(T-t)}{N}} - d}{u-d}$$

N-step call option

- Given a specific option, one can simplify the formula.
- For a call with K and stock $S(t)$, the payoff is $V(T) = \max[0, S(T) - K]$.
- Let n be the smallest number of steps such that $S(t)u^n - K \geq 0$.
- Then the value $C(t)$ of call option is

$$C(t) = S(t)F(n, N, \lambda) - e^{-r(T-t)} K F(n, N, \bar{\lambda})$$

  where F is computed using binomial distribution:

$$F(n, N, \lambda) = \sum_{i=n}^{N} \binom{N}{i} \lambda^i (1-\lambda)^{N-i}$$

  and $\bar{\lambda} = u e^{-r\frac{(T-t)}{N}} \lambda$.
- Given volatility σ, one usually sets $u = e^{\sigma\sqrt{T-t}}$ and $d = e^{-\sigma\sqrt{T-t}}$.
- In the limit $N \to \infty$ binomial pricing corresponds to Black-Scholes formula.

3 Replicating portfolio

Replicating portfolio

- We would like to buy X amount of stock at price $S(t)$ and invest B into a riskless bond to replicate the value V of an option:

$$V(t) = S(t)X + B$$

- At T the values change as follows

$$\begin{tikzcd} & V_u \\ V(t) \arrow[ur] \arrow[dr] & \\ & V_d \end{tikzcd} \qquad \begin{tikzcd} & Su \\ S(t) \arrow[ur] \arrow[dr] & \\ & Sd \end{tikzcd}$$

$$B \longrightarrow Be^{r(T-t)}$$

Replicating portfolio for one-step binomial

- Using $V_u = S(t)uX + Be^{r(T-t)}$ and $V_d = S(t)dX + Be^{r(T-t)}$ gives

$$X = \frac{V_u - V_d}{S(u-d)}\,, \qquad B = \frac{uV_d - dV_u}{e^{r(T-t)}(u-d)}$$

- Substituting into $V(t) = S(t)X + B$ gives the price of option:

$$V(t) = e^{-r(T-t)}[V_u\lambda + V_d(1-\lambda)]$$

  where $\lambda = \frac{e^{r(T-t)}-d}{u-d}$.

Reading

- Chapter 1, Sec. 1.3–1.5 (Elliott & Kopp, 2004).
- Chapter 5, 6 (Roman, 2012)
- Chapter 6, 8, Sec, 8.3.3 (Crack, 2014)

References

Crack, T. F. (2014). *Basic Black-Scholes: Option pricing and trading* (3rd ed.). Timothy Crack.

Elliott, R. J., & Kopp, P. E. (2004). *Mathematics of financial markets* (2nd ed.). Springer.

Roman, S. (2012). *Introduction to the mathematics of finance: Arbitrage and option pricing*. Springer.