Lecture 1: Options and rational pricing

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Contents

Notation

$\mathbb{N}$ — the set of natural numbers $\{1, 2, 3, \ldots\}$.

$\mathbb{R}$ — the field of real numbers $(-\infty, \infty)$.

t — time measured in years $t \in \mathbb{R}$.

T — a specific moment in time (e.g. the expiration time).

$x(t)$ — a value of $x \in X$ depending on time (i.e. $x(t)$ is a function of t).

$S(t)$ — stock price at $t \leq T$.

$V(t)$ — value of a derivative (e.g. an option) at $t \leq T$.

K — strike price at $t = T$.

r — anual rate of return.

1 Futures, forwards and options

Futures, forwards and options

Definition 1 (Forward Contract)**.** An *obligation* to buy (or to sell) stock S at $t = T$ for the *strike price* K.

Definition 2 (Option European (resp. American))**.** A contract giving the *right* (but not an obligation) to buy (or sell) S at $t = T$ (resp. $t \in [0, T]$) for K.

Call — an option to buy S.

Put — an option to sell S.

Long position means buying a contract (buyer).

Short position means selling a contract (seller / writer).

Definition 3 (Pricing problem)**.** Given information on stock price $S(t)$, strike price K, expiration time T, what is a fair price $V(t)$ of a contract (e.g. option or forward) at $t \leq T$?

Stock exchanges

- The largest is *Chicago Board of Options Exchange* (CBOE).
- An option series:

  `IBM JAN 50 CALLS`

  where $K = 50$, $T = Jan$.
- An example of a Call option symbol for IBM with $K = \$125$, $T =$ Jan 23, 2015:

  `IBM150123C00125000`

The purpose of options

Hedging

- An investment to protect from a risk of changing S.
- Example: Buying a protective put (*long put*) to protect from a price drop of S.

Speculation

- An investment to gain from a changing S.
- Example: Buying speclative call (*long call*) to gain from a price rise of S.

Question 1. *What would be the purpose of*

- *Selling a call (short call)?*
- *Selling a put (short call)?*

Leverage

- Suppose you have £1000 to invest.
- Let $S(t) = £100$ per share, so that you can buy 10 shares at t.
- If $S(T) = £105$ after $T - t = 1$ year, then your profit is $10 \times (S(T) - S(t)) = £50$ and the return

$$\frac{S(T) - S(t)}{S(t)} = \frac{£5}{£100} = 0.05$$

- Let the cost of a call option $C(t) = £4$ with $K = £100$, so that you can buy 250 calls at t.
- If $S(T) = £105$ after $T - t = 1$ year, then your profit is $250 \times (£S(T) - (K + C(t))) = £250$ and the return

$$\frac{S(T) - (K + C(t))}{C(t)} = \frac{£1}{£4} = 0.25$$

Question 2. *What if $S(T) = £95$?*

2 Terminal payoffs and payoffs curves

Terminal payoffs and payoffs curves

Long call :

$$V_{\text{call}}(T) = \max[0, S(T) - K]$$

Short call :

$$-V_{\text{call}}(T) = -\max[0, S(T) - K]$$

Long put :

$$V_{\text{put}}(T) = \max[0, K - S(T)]$$

Short put :

$$-V_{\text{put}}(T) = -\max[0, K - S(T)]$$

3 Riskless assets

Riskless assets

- Let r be the annual rate of return (APR) offered by a bank.
- If the dividents are compounded n times per year, then:

$$S(t+1) = \left(1 + \frac{r}{n}\right)^n S(t)$$

- Continuously compounded dividents

$$S(t+1) = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n S(t) = e^r S(t)$$

- Up to T years:

$$S(T) = e^{r(T-t)} S(t)\,, \qquad r = \frac{1}{T-t} \ln \frac{S(T)}{S(t)}$$

- The effective annual rate of return (EAR) is:

$$EAR = e^r - 1$$

4 No arbitrage and risk-neutral pricing

No arbitrage and market equilibrium

- The opportunity of generating riskless profit is *arbitrage*.
- The pricing theory assumes that the market offers *no arbitrage* opportunities.
- A market at an *equilibrium* (supply = demand) implies no arbitrage.

Risk-neutral pricing and arbitrage

Theorem 4. *No arbitrage implies that a forward price $F(t,T)$ at T of a stock with no dividents is*

$$F(t,T) = e^{r(T-t)} S(t)$$

where r is riskless APR, and $S(t)$ is the price at $t \le T$.

Proof.
- If $F(t,T) < e^{r(T-t)} S(t)$, then sell S at t, save and buy F at T.
- If $F(t,T) > e^{r(T-t)} S(t)$, then borrow to buy S at t, sell F and repay at T.

□

Current value of options

$$V(t) = \mathbb{E}\{e^{-r(T-t)} V(T)\} = e^{-r(T-t)} \mathbb{E}\{V(T)\}$$

where $V(T) = \max[0, S(T) - K]$ or $V(T) = \max[0, K - S(T)]$.

5 Call-put parity

Call-put parity

- Let $C(t)$, $P(t)$ denote prices at $t \leq T$ of call and put options respectively with strike price K.
- At expiration time T we have

$$C(T) - P(T) = \max[0, S(T) - K] - \max[0, K - S(T)] = S(T) - K$$

- And at $t \leq T$

$$C(t) - P(t) = S(t) - e^{-r(T-t)}K$$

- Therefore

$$C(t) + e^{-r(T-t)}K = S(t) + P(t)$$

- This means that call can be expressed using put and vice versa:

$$\begin{aligned} C(t) &= S(t) - e^{-r(T-t)}K + P(t) \\ P(t) &= e^{-r(T-t)}K - S(t) + C(t) \end{aligned}$$

- CBOE opened in 1973, and did not trade puts until 1977.

Reading

- Chapter 1, Sec. 1.1, 1.2 (Elliott & Kopp, 2004).
- Chapter 1, 2 (Roman, 2012)
- Chapter 1, 3 (Crack, 2014)

References

Crack, T. F. (2014). *Basic Black-Scholes: Option pricing and trading* (3rd ed.). Timothy Crack.

Elliott, R. J., & Kopp, P. E. (2004). *Mathematics of financial markets* (2nd ed.). Springer.

Roman, S. (2012). *Introduction to the mathematics of finance: Arbitrage and option pricing*. Springer.