Lecture 10: Introduction to logic

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Contents

1 Introduction

Historical Background

400 B.C. Aristotle's work provide foundation of two-valued logic: every proposition is either True or False (excluded middle).

1847 George Bool defined the calculus of deductive reasoning, which we now call *Boolean logic*.

1900 Jan Lukasiewicz proposed three-valued logic: True, False and Possible.

1912 Luitzen Brouwer laid foundations for intuitionistic logic.

1913 Bertrand Russell's work on foundations of mathematics.

1931 Kurt Gödel's *incompleteness* theorems.

1936 Garrett Birkhoff and John von Neumann introduced *quantum* logic.

1965 Lotfi A. Zadeh published '*Fuzzy Sets*' article.

Types of Logic

- Logic is a language of reasoning; a study of inference and reasoning.
- Knowledge is justified *true* belief.
- There exist different types of logic:

  Propositional logic of sentences or *propositions*.

  Predicate logic taking into account *quantities* (some, all).

Modal logic dealing with *possibilities* and *necessities*.

Fuzzy or multi-valued logic allowing for different degrees of truth.

Probabilistic logic for dealing with uncertainty.

Temporal logic for dealing with events in time.

Intuitionistic or constructive logic, rejecting the *law of the excluded middle*.

Quantum logic, rejecting the *distributivity law*.

- and many others.

2 Elements of Boolean Logic

Boolean logic (algebra)

- George Boole published his book *An Investigation of the Laws of Thought* in 1854 describing what we now call *Boolean logic* or *Boolean algebra*.
- It is an algebra on Boolean *variables* (having one of two values true or false) with Boolean *operations* (not, and, or).
- It was shown to be equivalent to the algebra of sets.

Boolean variables

Definition 1 (Boolean variable)**.** is any variable a, b that can have only two values:

$$a \in \{0, 1\}\,, \qquad b \in \{\texttt{False}, \texttt{True}\}$$

- Consider the following propositions:

  Pif is a dog   Dogs can fly

- Any fact, statement or *proposition* can be assumed (or believed) to be true or false, and so it can be considered as a Boolean variable.
- Two values are based on the *law of the excluded middle* due to Aristotle.

Remark 1 (Multi-valued logics)**.**
- *Jan Lukasiewicz proposed three-valued logic: True, False and Possible.*
- *In* fuzzy logic*, pioneered by Lotfi A. Zadeh, there is a continuum degree of truth.*

Boolean operations

Definition 2 (Boolean operation)**.** is a mapping from one or more Boolean variables to another.

- In algebra of numbers, an operation (e.g. $+$, $\times$) maps one or more numbers to another:

$$-1 \times 1 = -1\,, \qquad 3 + 1 = 4\,, \qquad 2 \times 2 = 4\,, \qquad 3 + 2 \times 4 = 11$$

- *Boolean logic* can be described as a complete system of Boolean functions that can be derived (or represented) using three elementary Boolean functions (operations):

  $\neg$ not (negation)

  $\wedge$ and (conjunction)

  $\vee$ or (disjunction)

- These operations are equivalent to set complement, set intersection ($\cap$) and set union ($\cup$) in the algebra of sets.

Negation (NOT, $\neg$)

- Let $a \in L$ be a Boolean variable $a \in L = \{0, 1\} \equiv \{\texttt{False}, \texttt{True}\}$

  Definition 3 ($\neg : L \to L$)**.**

a	$\neg a$
0	1
1	0

- The value of $\neg a$ is $1 - a$.

- Negation is equivalent to the complement of a set (the set of all elements $b \notin A$). If U is the universal set, and $A \subseteq U$, then the complement of A is

$$\bar{A} = U - A$$

Question 1 (Double negation)**.** *What is the value of* $\neg\neg a =$?

Conjunction (AND, $\wedge$)

Definition 4 ($\wedge : L \times L \to L$)**.**

a	b	$a \wedge b$
0	0	0
1	0	0
0	1	0
1	1	1

- The value of $a \wedge b$ is the *minimum* of $\{a, b\}$.
- Conjunction is equivalent to the intersection of sets

$$A \cap B$$

Question 2 (Law of contradiction)**.** *What is the value of*

$$a \wedge \neg a = ?$$

Disjunction (OR, $\vee$)

Definition 5 ($\vee : L \times L \to L$)**.**

a	b	$a \vee b$
0	0	0
1	0	1
0	1	1
1	1	1

- The value of $a \vee b$ is the *maximum* of $\{a, b\}$.
- Disjunction is equivalent to the union of sets

$$A \cup B$$

Question 3 (Law of the excluded middle)**.** *What is the value of*

$$a \vee \neg a = ?$$

Implication ($\to$, $\Rightarrow$)

Definition 6 ($\Rightarrow : L \times L \to L$)**.**

a	b	$a \Rightarrow b$
0	0	1
1	0	0
0	1	1
1	1	1

- The value of $a \Rightarrow b$ is the same as $\neg a \vee b$.
- Disjunction is equivalent to set inclusion

$$A \subseteq B$$

Question 4 (Inverse implication)**.** *Which of the implications below is equivalent to $a \Rightarrow b$?*

$$\neg a \Rightarrow \neg b \quad or \quad \neg a \Leftarrow \neg b$$

Equivalence ($\sim$)

Definition 7 ($\sim: L \times L \to L$)**.**

a	b	$a \sim b$
0	0	1
1	0	0
0	1	0
1	1	1

- The value of $a \sim b$ is the same as $(a \vee b) \wedge (\neg a \vee \neg b)$.
- Logical equivalence is the same as set equivalence:

$$A \subseteq B \quad \text{and} \quad A \supseteq B\,, \quad A \equiv B$$

Summary of Elementary Logical Operations

a	b	$\neg a$	$a \wedge b$	$a \vee b$	$a \Rightarrow b$	$a \sim b$
0	0	1	0	0	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
1	1	0	1	1	1	1

Duality (De Morgan's) laws :

$$\neg(a \wedge b) = \neg a \vee \neg b\,, \quad \neg(a \vee b) = \neg a \wedge \neg b$$

Absorption laws :

$$a \wedge (a \vee b) = a\,, \quad a \vee (a \wedge b) = a$$

Using Boolean Algebra

- Given values of some Boolean variables a, b, we can also infer the true or false value of propositions such as:

$$(a \wedge b) \vee (\neg a) \vee (\neg b) =?$$

- Suppose $a = 0$ and $b = 1$, then

$$\begin{aligned} &(0 \wedge 1) \vee (\neg 0) \vee (\neg 1) \\ &= \quad 0 \vee 1 \vee 0 \\ &= \quad 1 \end{aligned}$$

- The reasoning can be implemented on a digital computer (automated reasoning automated theorem proving systems).

Remark 2 (Boolean satisfiability)**.**

- *Checking if there exists an assignment of variables such that a proposition is true is called* Boolean satisfiability *problem (SAT).*
- *It is a classical example of an NP-hard problem.*

3 Applications of Logic in Knowledge Management

Use of Logic for Semantic Web

- Evaluating and applying rules.
- Inferring facts that were not stated explicitly.
- Providing explanations of facts or conclusions (using backward reasoning).
- Combining information from different sources.
- Detecting contradictions or conflicting statements.

Some Considerations

- The reasoning process and it outcome depends on the type of logic used (e.g. multi-valued, modal, temporal).
- Complexity can increase rapidly (exponentially).
- Incorrect facts or unreliable information can contaminate the knowledge base leading to wrong conclusions (knowledge pollution).

Additional Reading

1. Yang, Olson, and Kim (2004):

   > Comparison of first order predicate logic, fuzzy logic and non-monotonic logic as knowledge representation methodology

2. Saba (2007):

   > Language, logic and ontology: Uncovering the structure of commonsense knowledge

References

Saba, W. S. (2007). Language, logic and ontology: Uncovering the structure of commonsense knowledge. *International Journal of Human-Computer Studies*, *65*, 610-623.

Yang, K. H., Olson, D., & Kim, J. (2004). Comparison of first order predicate logic, fuzzy logic and non-monotonic logic as knowledge representation methodology. *Expert Systems with Applications*, *27*, 501-519.