Questions 8:
Game Theory

Roman Belavkin

Middlesex University

Question 1

Suppose of you have a choice between two lotteries A and B:

- Lottery A: The utility can have values -1, 0 or 1.
- Lottery B: The utility can have values -2, 0 or 2.

Suppose that all values have equal probabilities.

a) What choice does the maximum expected utility principle suggest?

b) Which of the lotteries has higher uncertainty (risk)?

Answer:

a) *The values of expected utilities are:*

$$\begin{aligned} E_A\{u\} &= -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = 0 \\ E_B\{u\} &= -2 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = 0 \end{aligned}$$

Because both expected values are equal, there is no maximum, and therefore the maximum expected utility principle cannot suggest any choice (i.e. we are indifferent between A and B).

b) *We can compute the variance of utility for each lottery:*

$$\begin{aligned} Var_A\{u\} &= (-1-0)^2 \cdot \frac{1}{3} + 0^2 \cdot \frac{1}{3} + (1-0)^2 \cdot \frac{1}{3} = \frac{2}{3} \\ Var_B\{u\} &= (-2-0)^2 \cdot \frac{1}{3} + 0^2 \cdot \frac{1}{3} + (2-0)^2 \cdot \frac{1}{3} = \frac{8}{3} \end{aligned}$$

So, lottery B is more risky and more uncertain.

Question 2

Describe what is a payoff matrix in a zero-sum 2-person game. Give example.

Answer: *There are two players, A and B, in a 2-person game. Each has a set of strategies $S^A = \{s_1^A, \ldots, s_m^A\}$ and $S^B = \{s_1^B, \ldots, s_m^B\}$. The payoff matrix for, say, player A is the $m \times n$ matrix $u_A = (u_{ij})$, where u_{ij} is the utility value that player A receives if player A chooses strategy s_i^A and player B chooses strategy s_j^B. In a zero-sum 2-person game, $u_B = -u_A$ so that $u_A + u_B = 0$.*

An examples are the Rock Paper Scissors, the Penny Matching games. Their payoff matrices for player A are respectively:

$$u_A = \begin{pmatrix} u_{rr} & u_{rp} & u_{rs} \\ u_{pr} & u_{pp} & u_{ps} \\ u_{sr} & u_{sp} & u_{ss} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$u_A = \begin{pmatrix} u_{hh} & u_{ht} \\ u_{th} & u_{tt} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Also, many classical games, such as Chess, are zero-sum.

Question 3

What is a mixed strategy? What is an optimal mixed strategy?

Answer: *Let $S^A = \{s_1^A, \ldots, s_m^A\}$ be the set of m strategies of player A in a 2-person game. A mixed strategy of player A is a probability distribution P^A over the set S^A. That is, a mixed strategy assigns a probability p_i^A to each strategy s_i^A.*

An optimal mixed strategy for player A in a 2-person game is such $\bar{P}^A$ that maximises the minimum of the expected utility, taken over all mixed strategies of player B:

$$\max_{P^A} \min_{P^B} E_{P^A P^B}\{u_A\}$$

Question 4

Suppose that players A and B play the Rock Paper Scissors game (paper wins over rock, scissors win over paper, rock wins over scissors, and draw for any matching pair). Denote by $S^A = S^B = \{r, p, s\}$ the sets of strategies for both players in each game, which correspond to rock, paper and scissors respectively. Suppose that player A uses mixed strategy $P^A = \{0.1, 0.2, 0.7\}$, where each number is the probability of rock, paper or scissors respectively. Suggest a winning mixed strategy for player B. Use the expected payoff to prove that the strategy is winning.

Answer: *Player A chooses mostly scissors, because $p_s^A = 0.7$. Thus, to win, player B should choose mostly rock. For example, $P^B = \{1, 0, 0\}$ should do the job. If the values of payoff to player B are $\{-1, 0, 1\}$ corresponding to loosing, draw and winning, then the expected payoff is*

$$E_{P^A P^B}\{u_A\} = -1 \cdot 0.3 \cdot 1 + 0 \cdot 0.1 \cdot 1 + 1 \cdot 0.7 \cdot 1 = 0.4$$

which is a positive expected payoff to player B.