Lecture 2: Choice and Optimisation

Dr. Roman V Belavkin

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Contents

1 Choice and Preference Relation

AI and Decision-Making

- Recall some definitions of intelligent systems:

  '. . . automation of deductive constructions'

  '. . . study and design of intelligent agents'

- What is an autonomous agent?
- What is an intelligent agent?

Question 1. *Consider an computer program that can 'play' chess by moving randomly the figures.*

1. *Is it autonomous?*
2. *Is it intelligent?*
3. *What is different between playing chess and simply moving the figures randomly?*

Choice Problems and Choice Sets

- To understand what is decision-making we first need to understand what is making a simple choice, and why one choice is different from another.
- Consider the following examples of sets. What is your choice? Why?

1.
$$\left\{\text{Apple}, \text{Orange}\right\}$$

2.
$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

3.
$$\{£1, £2, £3, £4, £5, £6, £7, £8, £9, £10\}$$

4.
$$\{-£1, -£2, -£3, -£4, -£5, -£6, -£7, -£8, -£9, -£10\}$$

5.
$$\{\$303, £202\}$$

Preference Relation

- Let Ω be our choice set.
- Every pair of elements in the choice set must be comparable.
- Given a pair a, $b \in \Omega$ you should be able to choose one.

  Definition 1 (Preference relation)**.** $\lesssim$ is a binary relation $\lesssim \subseteq \Omega \times \Omega$ that is

  1. *Total*: either $a \lesssim b$ or $b \lesssim a$ for all a, $b \in \Omega$.
  2. *Transitive*: If $a \lesssim b$ and $b \lesssim c$, then $a \lesssim c$

- We shall assume that choice sets are sets with preference relation $(\Omega, \lesssim)$

Equivalence and Order

- Aristotle described a man who was as hungry as he was thirsty and placed exactly between food and water, and therefore he did not move.
- Let Ω be a choice set, that is a set with preference relation $\lesssim$.
- If $a \lesssim b$ and $b \lesssim a$, then a and b are related symmetrically, or a and b are preferred 'equivalently':

$$a \sim b := a \lesssim b \text{ and } b \lesssim a$$

- Relation $a \sim b$ is *equivalence* or sometimes called *indifference*.
- If $a \lesssim b$, but not $b \lesssim a$ (i.e. $a \nsim b$), then b is preferred strictly to a.
- We can simply write

$$a < b := a \lesssim b \text{ and not } b \lesssim a$$

- Note that $a < b$ are related anti-symmetrically, and such relation is also called *order*.

Optimal Choice

- Let $\Omega = \{£5, £10, £20\}$
- Then the optimal choice is obviously

$$£20 = \max\{£5, £10, £20\}$$

Definition 2 (Greatest element (maximum, top)). of set Ω with $\lesssim$ is $\top \in \Omega$ such that $a \lesssim \top$ or all $a \in \Omega$.

- *Optimisation* is a procedure of finding the top element in the choice set.

Remark 1. *In a choice set, there can be several top elements. This is because $a \sim b$ does note mean $a = b$.*

Remark 2. *In mathematics, we can think of sets, in which the top element does not exist (e.g. the set of real numbers $\mathbb{R}$).*

Optimisation with Constraints

- When we solve optimisation problems, often there are additional conditions, called *constraints*.

  Example 3.

$$\begin{array}{cl} \text{Maximise} & \Omega = \{1, 2, 3, 4, 5, 6\} \\ \text{Subject to} & a \in \Omega \text{ is prime number} \end{array}$$

- Elements $a \in \Omega$ satisfying the constraint is are called *feasible*.
- Feasible elements form a subset of the initial choice set:

$$\{a \in \Omega : \text{subject to } a \dots\} \subseteq \Omega$$

Question 2. *Can constraints 'improve' the solution?*

2 Utility Function

The Utility Function

- Sets with preference relation (choice sets) are very similar to ordered sets (in fact, they are *pre-ordered*):

$$\{\dots, -2, -1, 0, 1, 2, 3, \dots\}$$

Definition 4. *Utility* representation of $(\Omega, \lesssim)$ is any numerical function $u : \Omega \to \mathbb{R}$ such that

$$a \lesssim b \quad \text{if and only if} \quad u(a) \le u(b)$$

- In other words, utility assigns a number (priority, rank) to each element of the choice set such that the more an element is preferred the higher is the number.

Example 5. Two job offers: Job1 with salary $£18K$ and Job2 with $£30K$. Clearly,

$$\text{Job1} \lesssim \text{Job2 because } £18,000 \le £30,000$$

Examples of Utility Functions

Example 6 (Binary Utility). When all elements in the choice set Ω can be divided into two classes (e.g. Failure $\lesssim$ Success), then we can use $u : \Omega \to \{0, 1\}$:

$$u(\omega) = \begin{cases} 1 & \text{if } \omega \text{ is a success} \\ 0 & \text{otherwise} \end{cases}$$

Example 7 (Ternary Utility). If there is a third (e.g. neutral) element, then we can use $u : \Omega \to \{-1, 0, 1\}$ (e.g. recall the *Rock-Paper-Scissors* game).

Example 8 (Multivalued Utility). If there are many classes in the choice set, then the utility must have just as many values $u : \Omega \to [a, b] \subset \mathbb{R}$ (e.g. reflecting different profits, speeds, etc).

Optimisation as Maximisation of Utility

- The optimal element is the greatest (top) element $\top \in \Omega$.
- Thus, it corresponds to the greatest (maximum) value of utility:

$$u(\top) = \max u(x) \ge u(x) \quad \text{for all } x \in \Omega$$

- Conversely, the maximum of utility corresponds to the top element:

$$\top = u^{-1}(\max u)$$

where u^{-1} denotes the inverse of the utility function.

- This is sometimes written as

$$\top = \arg\max u(x)$$

where arg stand for 'argument'.

Duality of Utility and Cost

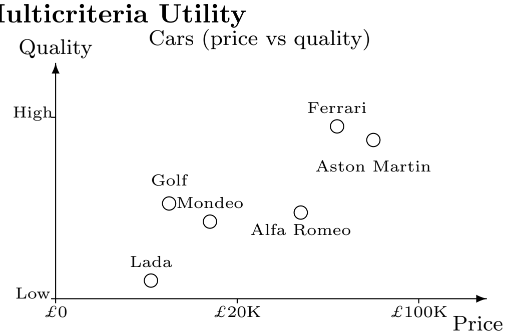
- Often we need to minimise the cost $c : \Omega \to \mathbb{R}$ of elements in the choice set Ω:
$$a \lesssim b \quad \text{if and only if} \quad c(a) \geq c(b)$$
- Notice that
$$c(a) \geq c(b) \iff -c(a) \leq -c(b)$$
(i.e. $3 \geq 2$ is equivalent to $-3 \leq -2$)
- Minimisation of cost is equivalent to maximisation of negative cost. Or dually, maximisation of utility is equivalent to minimisation of negative utility.
- Utility and negative cost are equivalent:
$$\text{Utility} \equiv -\text{Cost}$$

3 Multicriteria Decision Making

Multicriteria Decision Making

- Recall the example with two job offers: Job1 salary $£18K$; Job2 with $£30K$.
- Let Job1 be in City1 and Job2 be in City2. Suppose now that City1 is located near the sea, has a good climate, nice restaurants, cheap food and let City2 has none of these.
- What will be your preference?
- How many objectives you considered?
- Has learning new information changed your decision?

Multicriteria Utility



- We could use the following utility:

$$u(x) = \frac{\text{Quality}(x)}{\text{Price}(x)}$$

- Better still

$$u(x) = \text{Quality}(x) - \text{Price}(x)$$

- Additive multicriteria utility:

$$u(x) = u_1(x) + u_2(x)$$

  where

$$\begin{aligned} u_1(x) &= \text{Quality}(x) \\ u_2(x) &= -\text{Price}(x) \end{aligned}$$

Weighted Criteria

- Consider additive multicriteria utility:

$$u(x) = u_1(x) + u_2(x) + \cdots + u_n(x) = \sum_{i=1}^{n} u_i(x)$$

- If some criteria are more important than others, then they can be multiplied by coefficients $w_i \in \mathbb{R}$ (weights):

$$u(x) = w_1 u_1(x) + w_2 u_2(x) + \cdots + w_n u_n(x) = \sum_{i=1}^{n} w_i u_i(x)$$

- The weights represent a preference relation on the set of criteria.

Critique of the Rational Approach

- There are many problems with the idea of optimal choice and rationality.
- People are known to make choices which can be described as *suboptimal*, but satisfactory.
- To understand this, recall optimisation with constraints (e.g. time constraints to make decisions, constraints on resources, information constraints, motivation constraints, etc).
- A decision can be seen as suboptimal simply because we do not appreciate all the constraints that a decision-maker had to take into account.

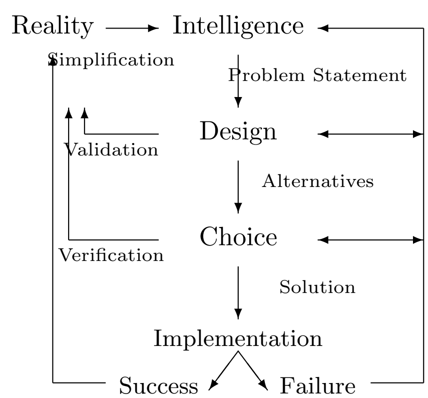
4 Types and Phases of Decisions

The Structure of Decisions

Herbert Simon introduced the idea of **structured** (programmable) and **unstructured** (nonprogrammable) decisions.

Structured	Semi–structured	Unstructured
goals defined	···	the outcomes are uncertain
procedures are known	···	appear in unique context
information is obtainable and manageable	···	the resources are hard to assess

The Three Phases of Decision Making



Simon (1977)

References