Mutation and Optimal Search of Sequences in Nested Hamming Spaces

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Evolution as an Information Dynamic System

- EPSRC Sandpit '*Math of Life*' (July, 2009):

- Three year project (2010–12)

  Middlesex University : Roman Belavkin
  University of Warwick : John Aston
  University of Keele : Alastair Channon & Elizabeth Aston
  University of Manchester : Chris Knight & Rok Krasovec

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Evolution : search for optimal codes:

$$\top_\theta := \bigvee_\theta \Omega_l$$

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Question

Rugged landscape?

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Definition (Relatively Monotonic Landscape)

f is locally monotonic (isomorphic) relative to a metric d, if there exist $B(\top, l) := \{\omega : d(\top,\omega) \leq l\}$, $\top = \sup \Omega$, such that $\forall\, a, b \in B(\top, l)$:

$$-d(\top,a) \leq -d(\top,b) \quad \Longrightarrow (\iff) \quad f(a) \leq f(b)$$

Point Mutation

Let $a \mapsto b \in \mathcal{H}^l_\alpha$ and denote $q := P(a)$, $p := P(b)$.

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Variational problem

- Linear programming problem:

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Proposition

Solution to the variational problem is achieved by independently mutating each letter with probability $\mu = \upsilon/l$. For $l \to \infty$ this corresponds to Poisson process with mutation rate υ.

Mutation and Adaptation in a Hamming Space

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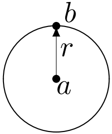

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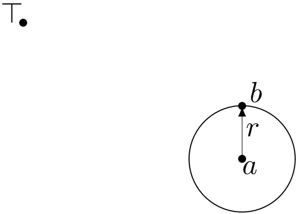

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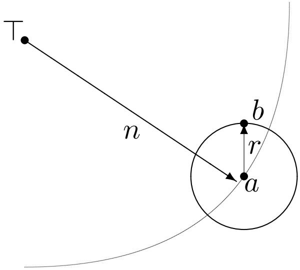

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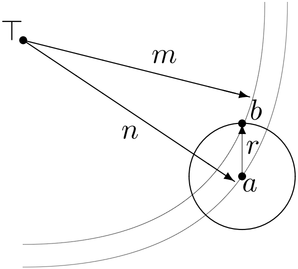

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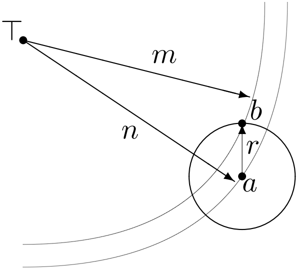

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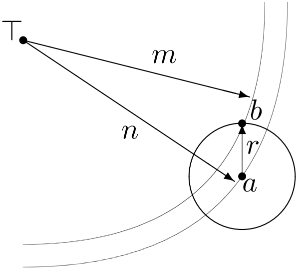

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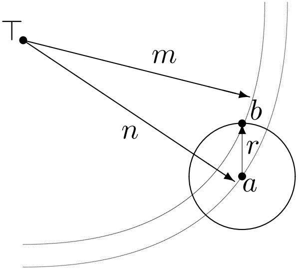

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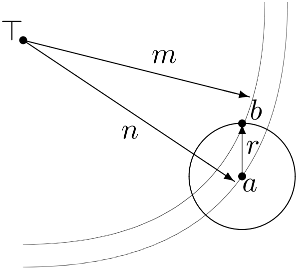

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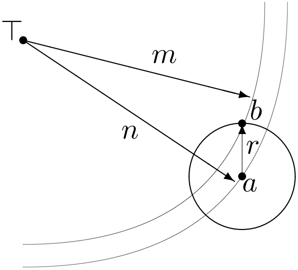

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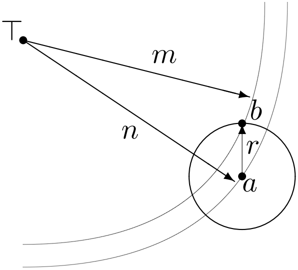

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- $\mu(x)$ — control function, $T_{\mu(x)}$, $\mathbb{E}_{\mu(x)}\{x_{s+t}\}$.

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Instantaneous

- Maximum adaptation in no more than λ generations

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Adaptation in One Generation

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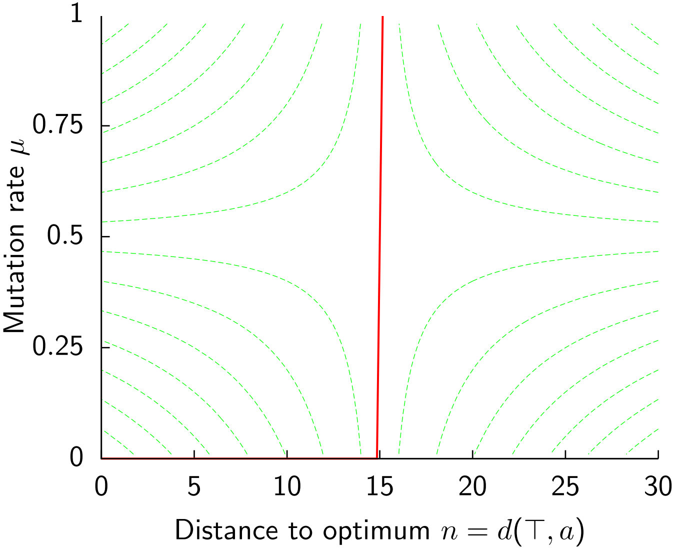
- In this case the optimal function is

$$\mu(n) := \begin{cases} 0 & \text{if } n < l(1 - 1/\alpha) \\ \frac{1}{2} & \text{if } n = l(1 - 1/\alpha) \\ 1 & \text{otherwise} \end{cases}$$

Step function



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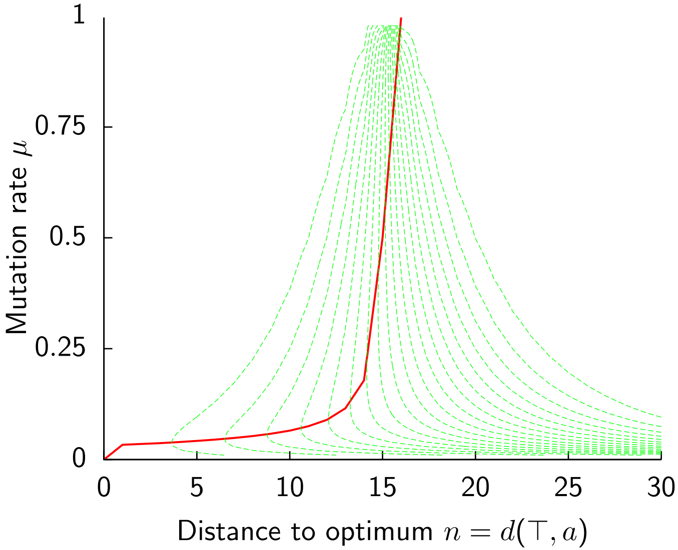
- This corresponds to maximization of $\mathbb{E}\{u(m,n)\}$, where

$$u(m,n) := \begin{cases} 1 & \text{if } m < n \\ 0 & \text{otherwise} \end{cases}$$

max $P_{\mu}(m < n \mid n)\}$



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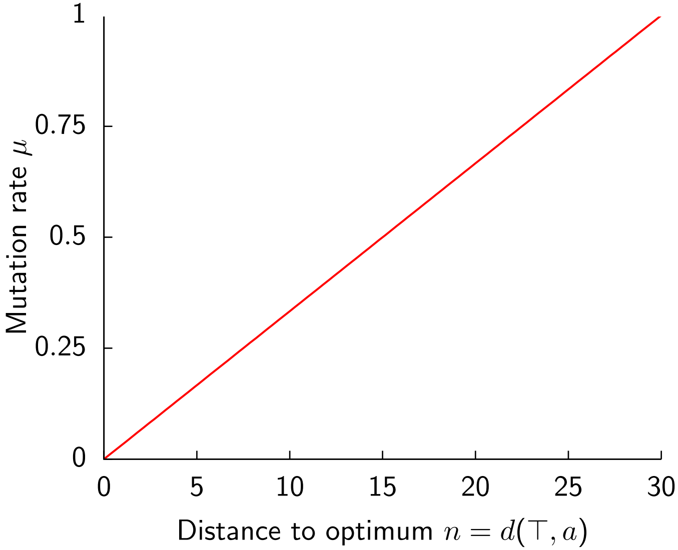
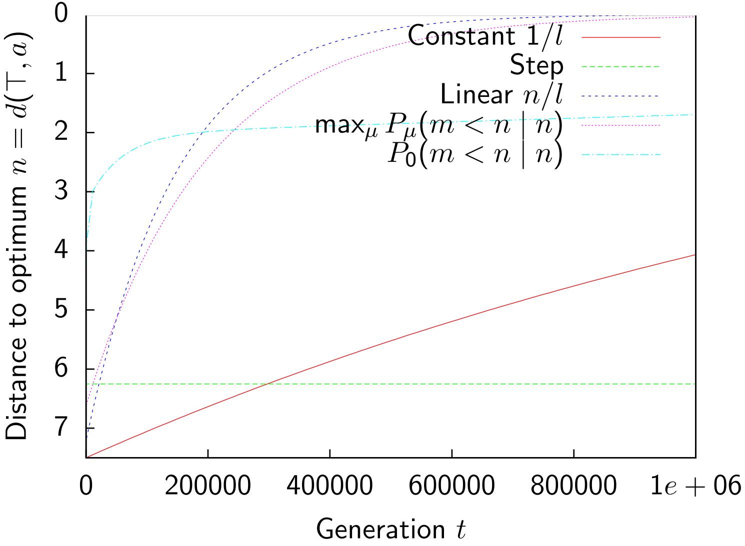
- *Optimal for Boolean landscapes (Needle in a haystack).*

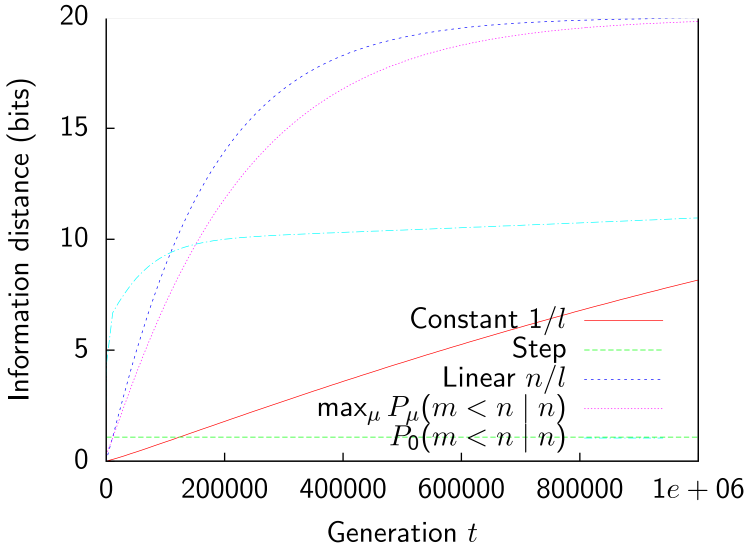
Linear function



Evolution of Fitness in Time



Information Divergence in Time



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Optimal Evolution in Information

Information Dynamics (Belavkin, 2010, 2011)

- Maximum adaptation in no more than λ bits between p_s and p_{s+t}:

$$\text{maximize} \quad \mathbb{E}_{\mu(x)}\{x_{s+t}\} \quad \text{subject to} \quad \mathbb{E}\{\log(p_{s+t}/p_s)\} \leq \lambda$$

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Information Heuristics $t \leq \lambda \iff I_{KL}(p_{s+t}, p_s) \leq \lambda$

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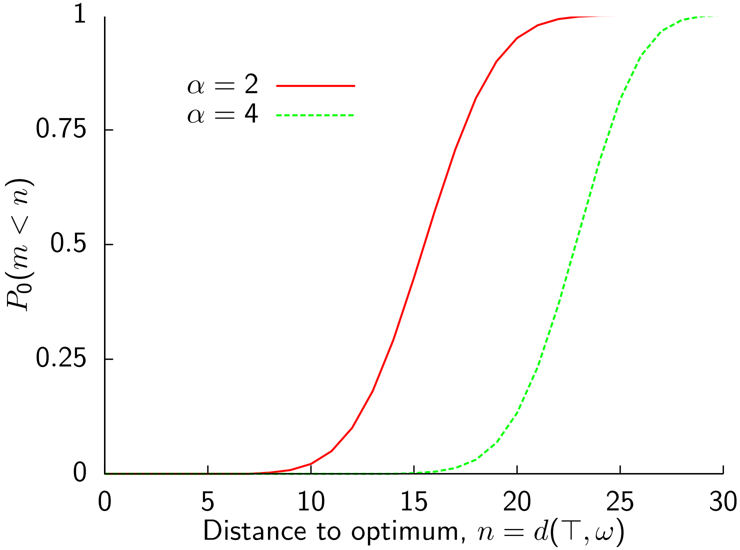
Informed Mutation Rate

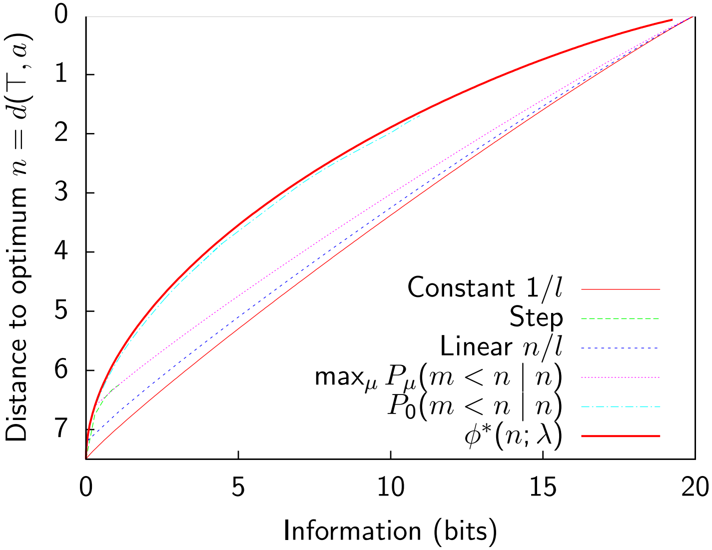
In a (weakly) monotonic landscape we can use CDF of empirical frequency P_e of observed fitness values:

$$P_0(m) \iff P_e(x) \quad \text{and} \quad P_0(m < n) \iff P_e(x_r > x)$$

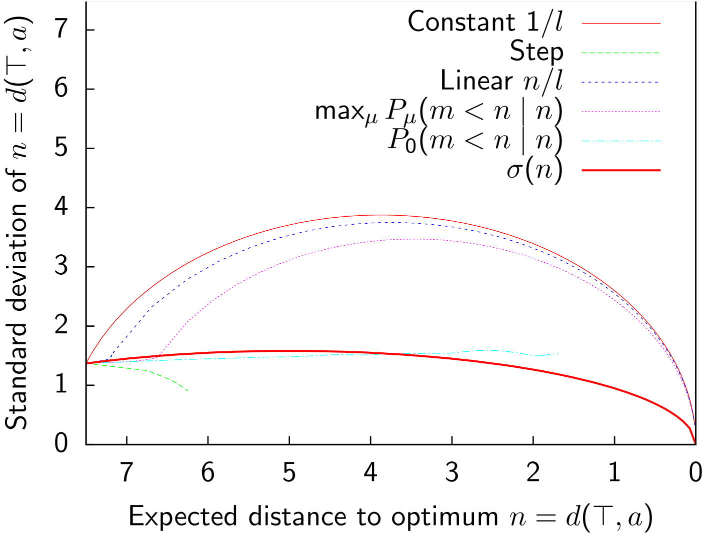
Minimal Information Control



Evolution of Fitness in Information



Fitness Variance and Expectation



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- *Have biological organisms evolved such controls?*
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