Optimal measures and transition kernels

Roman V. Belavkin

Middlesex University

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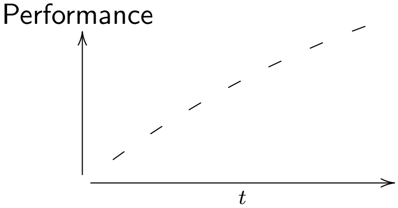

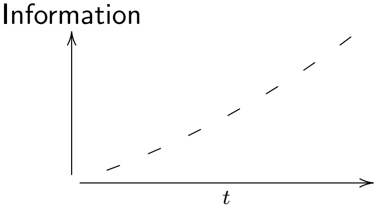

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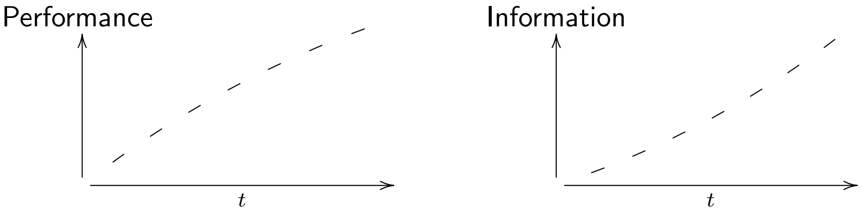

- In practice, stochastic and non-deterministic algorithms are generally more successful (e.g. simulated annealing, genetic algorithms).

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- *Y is a module over $X \subset Y$ with* $\|y\|_1 = |\langle 1, y\rangle|$

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- A number of properties: Rao-Cramer, maximum entropy.

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$$dp_\beta(a \mid b) = e^{\beta x(a,b) - \Psi(\beta,b)}\, dp_0(a) \to \delta_{f(b)}(a) \quad \text{as } \beta \to \infty$$

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Remark

$\mathbb{E}_p\{x\}$ ($\langle x, \cdot \rangle$) is a unique functional such that $\lesssim$ on $\mathcal{P}(\Omega)$ (Y) is compatible with the linear structure of Y and Archimedean.

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Example (Relative information)

$$F_{KL}(y) := \begin{cases} \left\langle \ln \frac{y}{y_0}, y \right\rangle - \langle 1, y - y_0 \rangle & \text{if } y > 0 \text{ and } y_0 > 0 \\ \langle 1, y_0 \rangle & \text{if } y = 0 \text{ and } y_0 > 0 \\ \infty & \text{otherwise} \end{cases}$$

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  with $\overline{x}(\lambda) := -\infty$ if $\lambda < \inf F$, and $\exists\, x : \overline{x}(\infty) = \infty$.
- Function $\overline{x}(\lambda)$ has inverse

$$\overline{x}^{-1}(\upsilon) := \inf\{F(y) : \langle x, y\rangle \geq \upsilon\}$$

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Element y_β *solves* $\overline{x}(\lambda) := \sup\{\langle x, y\rangle : F(y) \leq \lambda\}$ *iff:*

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3. Sufficient by convexity of F^{**}.

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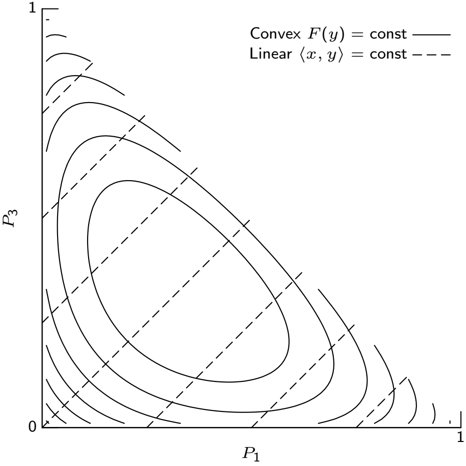
- If $x(a,b) = -\frac{1}{2}(a-b)^2$, a,$b \in \mathbb{R}^1$, p_β is Gaussian with $\sigma^2 = \beta^{-1}$.

Geometric Interpretation



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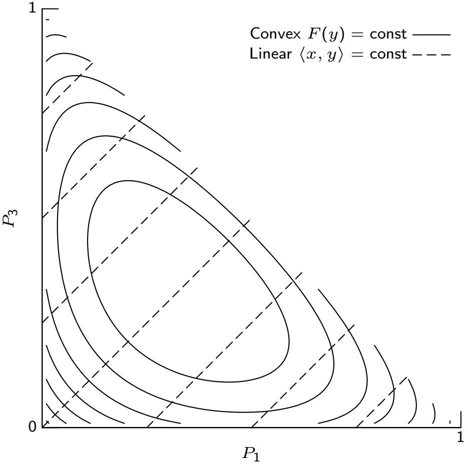

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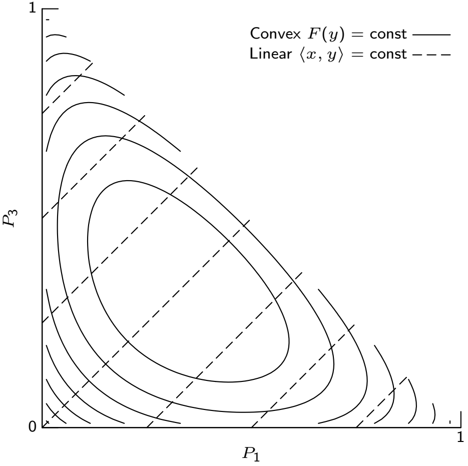

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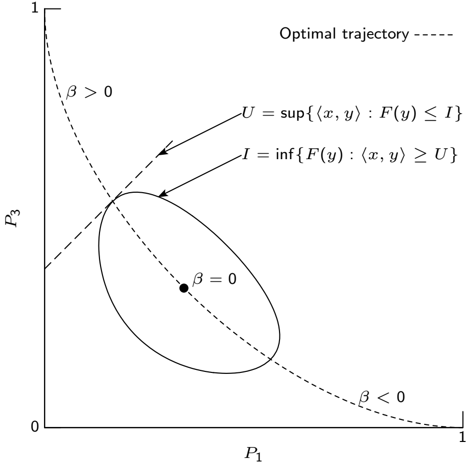

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- If F is strictly convex, then y_β is unique.

Optimal Value Function $\overline{x}(\lambda)$

Proposition

Function $\overline{x}(\lambda) := \sup\{\langle x, y\rangle : F(y) \leq \lambda\}$ *is concave and strictly increasing for* $\lambda \in (\inf F, F(\delta_x))$.

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- Then the restriction of F^* to M and its dual are

$$F^*(x)|_M = F^*(P_M x) \qquad F^{**}([y]) := \inf\{F^{**}(y) : y \in [y]\}$$

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- $\forall\,\lambda \in [\inf F, \sup F]\ \exists\, y_\beta \in \{y_\beta\}_x : y_\beta(M) = 0$
- Construct $\{y^\circ_\beta\}_x$: $M = \sup\{M' \subset X : \exists\, y^\circ_\beta \in \{y_\beta\}_x,\ y^\circ_\beta(M') = 0\}$

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- Hence

$$\partial F(\cdot) = \ln(\cdot) \quad \text{and} \quad \partial F^*(\cdot) = \exp(\cdot)$$

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Corollary

Let $p_\beta \in \mathcal{P}(A \times B)$ *be solutions* $\forall\, \lambda = F(p)$. *Let* F^* *be strictly convex.*
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Then p_β *is deterministic if and only if* $\lambda \geq F(\delta_x)$ *or* $\langle x, p_\beta \rangle = \langle x, \delta_x \rangle$*.*

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Optimal Transition Kernels

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($\Leftarrow$) obvious.

Strict Inequalities

Corollary

Let $p_\beta \in \mathcal{P}(A \times B)$ *be solutions* $\forall\, \lambda = F(p)$*. Let* F^* *be strictly convex.*
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$$\begin{aligned} \langle x, p_f \rangle < \langle x, p_\beta \rangle &\qquad F(p_f) = F(p_\beta) \in (\inf F, F(\delta_x)) \\ F(p_f) > F(p_\beta) &\qquad \langle x, p_f \rangle = \langle x, p_\beta \rangle \in (\overline{\upsilon}_0, \langle x, \delta_x \rangle) \end{aligned}$$

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Proof.

Based on inequalities (equalities $\iff$ $p_\beta \in \partial F^*(\beta x)$):

$$\begin{array}{rl} \langle x, y\rangle \leq F^*(x) + F(y)\,, & \quad F(y) \geq \langle x, y\rangle - F^*(x) \\ \beta\langle x, p_\beta\rangle = F^*(\beta x) + F(p_\beta)\,, & \quad F(p_\beta) = \beta\langle x, p_\beta\rangle - F^*(\beta x) \end{array}$$

$\square$

Example: Motivation

- Construct an example, where $\exists\,\lambda \in (\inf F, F(\delta_x))$ or $\upsilon \in (\overline{\upsilon}_0, \langle x, \delta_x\rangle)$ and for any deterministic p_f

$$\begin{aligned} \mathbb{E}_{p_f}\{x\} &= -\infty \qquad \text{if } I(p_f, p_0) \leq \lambda \\ I(p_f, p_0) &= \infty \qquad \text{if } \mathbb{E}_{p_f}\{x\} \geq \upsilon \end{aligned}$$

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- Strict inequalities $\langle x, p_f\rangle < \langle x, p_\beta\rangle$, $F(p_f) > F(p_\beta)$ of Corollary 6 would imply that for non-deterministic solution p_β

$$\begin{aligned} \mathbb{E}_{p_\beta}\{x\} &> -\infty \qquad \text{and } I(p_\beta, p_0) \leq \lambda \\ I(p_\beta, p_0) &< \infty \qquad \text{and } \mathbb{E}_{p_\beta}\{x\} \geq \upsilon \end{aligned}$$

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$$\begin{aligned} F_S(p_\beta) &= \int_A dp(a) \int_B \ln \frac{dp(b \mid a)}{dp(b)}\, dp(b \mid a) \\ &= \int_A dp(a) \int_B \left\{ \beta\, x(a,b) - \ln \int_B e^{\beta\, x(a,b)}\, db - \ln \frac{dp(b)}{db} \right\} dp(b \mid a) \\ &= \beta\, \mathbb{E}_{p_\beta}\{x\} - \Psi_0(\beta) + H\{p(b)\} \end{aligned}$$

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Optimal Mean-Squared Communication

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- If $dp(b) = [\pi(b^2 + 1)]^{-1}\, db$ (Cauchy distribution), then

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$$\begin{aligned}
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&= -\frac{1}{2}\frac{\sqrt{2\pi\beta^{-3}}}{\sqrt{2\pi\beta^{-1}}}\int_B dp(b) = -\frac{1}{2}\beta^{-1}
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- Lagrange multiplier β^{-1} is related to $F_S(p) = \lambda$

$$\beta = 2\pi e^{1-2[H\{p(b)\}-\lambda]}$$

Optimal Non-Deterministic Kernel

- Take Shannon information $F_S(p) = I_{KL}(dp(a,b), dp(a)\, dp(b))$.
- Then $dp_\beta(a \mid b)$ is exponential, and for $x(a,b) = -\frac{1}{2}(a-b)^2$

$$dp_\beta(a \mid b) = \frac{e^{\beta\, x(a,b)}\, da}{\int_A e^{\beta\, x(a,b)}\, da} = \frac{1}{\sqrt{2\pi\beta^{-1}}}\, e^{-\frac{1}{2}\beta(a-b)^2}\, da$$

- The expected utility is

$$\begin{aligned} \mathbb{E}_{p_\beta}\{x\} &= -\frac{1}{2}\frac{1}{\sqrt{2\pi\beta^{-1}}}\int_B dp(b) \int_{-\infty}^{\infty} (a-b)^2\, e^{-\frac{1}{2}\beta(a-b)^2}\, da \\ &= -\frac{1}{2}\frac{\sqrt{2\pi\beta^{-3}}}{\sqrt{2\pi\beta^{-1}}}\int_B dp(b) = -\frac{1}{2}\beta^{-1} \end{aligned}$$

- Lagrange multiplier β^{-1} is related to $F_S(p) = \lambda$

$$\beta = 2\pi e^{1-2[H\{p(b)\}-\lambda]}$$

- $\mathbb{E}_{p_\beta}\{x\} > -\infty$ if $H\{p(b)\} - \lambda < \infty$ or $\beta > 0$.

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- Then $\mathbb{E}_{p_f}\{x\} = -\infty$ for all $F(p_f) < \sup F$.

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- No contradiction with theory of optimal statistical decisions.

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