Fixed-parameter tractability and approximation via linear programming

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Habilitationsvotrag
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Outline

- Basics: Dealing with hard problems
- The usual suspect: Vertex Cover
- An interesting case: Independent set of (pseudo-) disks
- A good problem: Scheduling on identical machines

Dealing with NP-hard problems

Approach I: Approximation Algorithms

'...sucrifice optimality for polynomiality...'

Let Π be an optimization problem (with an objective function). Let A be an algorithm that returns, for every instance I, a feasible solution. Let $\alpha : \mathbb{Z}^+ \to \mathbb{Q}^+$.

A is a α-approximation algorithm for Π if

- there exists a polynomial p such that for every instance I, the runtime of A on I is bounded by $p(|I|)$,
- for every instance I, A produces a solution whose value is within a factor of $\alpha(|I|)$ of the value of an optimal solution for I. ($\alpha > 1$ for min, $\alpha < 1$ for max)

Approach I: Approximation Algorithms

'...sucrifice optimality for polynomiality...'
'...but get as close as you want...?'

Polynomial-time approximation scheme (PTAS):

A family of algorithms $\{A_\epsilon\}$, such that, for each $\epsilon > 0$, A_ϵ is a $(1+\epsilon)$-approximation algorithm (for min) or a $(1-\epsilon)$-approximation algorithm (for max).

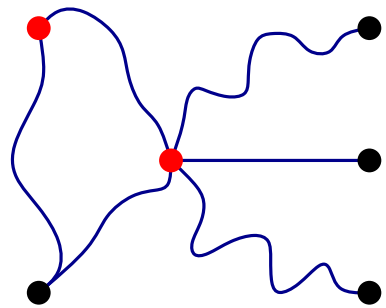
Instance I and $\epsilon > 0$:

PTAS: running time is $|I|^{f(1/\epsilon)}$
EPTAS (Efficient PTAS): running time is $f(1/\epsilon) \cdot |I|^{O(1)}$
FPTAS (Fully PTAS): running time is $(1/\epsilon)^{O(1)} \cdot |I|^{O(1)}$

Approach I: Approximation Algorithms

Example: Minimum Vertex Cover

Given a graph $G(V, E)$, find a minimum-size vertex cover, i.e., a set $V' \subseteq V$ such that every edge in E is incident to at least one vertex in V'.



- 2-approximable [Nemhauser and Trotter '75], and many others..
- no α-approximation algorithm exists for $\alpha < 1.36$..
  (unless P=NP) [Dinur and Safra '02]
- EPTAS for planar graphs [Baker '94]

Approach II: Fixed-parameter tractability

'...aim for exact solution...
...express the running time as a function of the input size and some parameter of the input...confine exponentiality in the parameter...'

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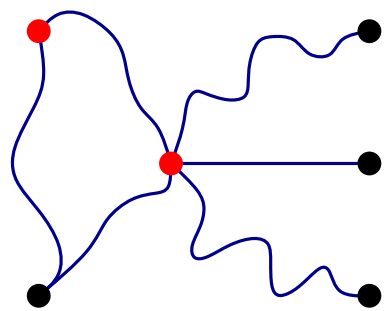
Vertex Cover (decision version)

Given a graph G with n vertices, and an integer k, decide whether G has a vertex cover of size k.



Solvable

- in $O(kn^{k+1})$ time (complete enumeration)

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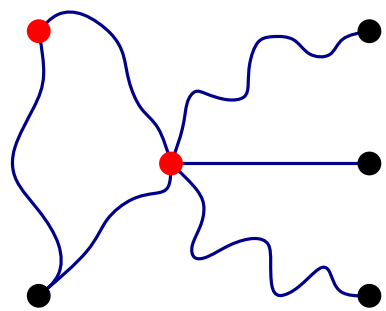


Solvable

- in $O(kn^{k+1})$ time (complete enumeration)
- but also in $O(2^k \cdot kn)$ time: fixed-parameter tractable w.r.t k (...and also in $O(1.2738^k + kn)$ time[Chen, Kanj, Xia '10])

Approach II: Fixed-parameter tractability

A parameterization of a decision problem Π is a function that assigns an integer parameter k to each instance I of Π.

A parameter can be

- explicit in the input (e.g., the size of the cover k in the input (G, k) of Vertex Cover)
- implicit in the input (e.g., the maximum degree of the input graph G)

A parameterized problem is *fixed-parameter tractable* if it can be solved in $f(k) \cdot |I|^{O(1)}$ time, where f is a computable function depending only on k.

FPT: The class of all fixed-parameter tractable problems

Approach II: Fixed-parameter tractability

The *standard parameterization* of an optimization problem is the 'natural' parameterization (i.e., with respect to the solution size) of the corresponding decision problem.

Vertex Cover

Input	:	(G, k)
Parameter	:	k
Question	:	Is there a vertex cover of size k ?

Fact: If an optimization problem has an EPTAS then its standard parameterization is in FPT [Bazgan '95]

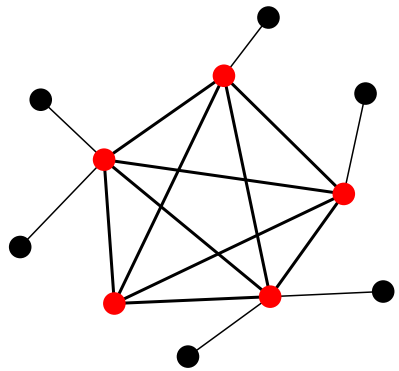
(The converse is not true, e.g., Vertex Cover)

Approach II: Fixed-parameter (in)tractability

'...are there problems for which there is evidence that they are not in FPT...?'

CLIQUE

Input : (G, k)
Parameter : k
Question : Is there a clique of size k ?
(clique: a complete subgraph)



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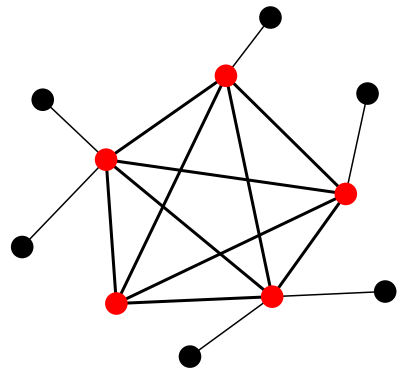

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CLIQUE is W[1]-complete [Downey and Fellows '95]

Simply put: CLIQUE is not in FPT, under standard assumptions

Also: A problem that is W[1]-hard has no EPTAS

Tools: Linear and Integer Linear Programming

LP:

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ \text{where} \quad & x \in \mathbb{R}^n,\ A \in \mathbb{Q}^{m \times n},\ b \in \mathbb{Q}^m,\ c \in \mathbb{Q}^n \end{aligned}$$

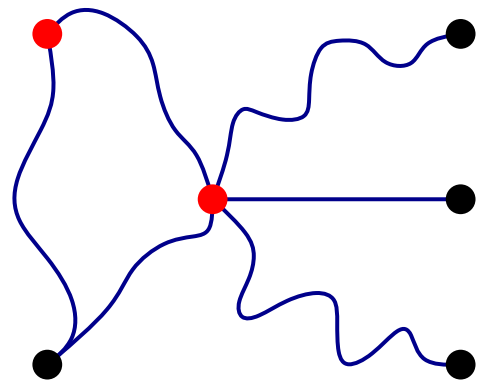
Fact: n-variable LP can be solved in $O(n^3 L)$ arithmetic operations and space polynomial in L, where L is the input length.
[Karmarkar '84, Ye '91]

ILP-F:

$$\begin{aligned} \text{Input} \quad & : A \in \mathbb{Z}^{m \times n},\ b \in \mathbb{Z}^m \\ \text{Parameter} \quad & : n \ (\#\ \text{variables}) \\ \text{Question} \quad & : \text{Is there } x \in \mathbb{Z}^n \text{ such that } Ax \leq b\ ? \end{aligned}$$

Fact: n-variable ILP-F can be solved in $O(n^{2.5n+o(n)} L)$ arithmetic operations and space polynomial in L, where L is the input length.
[Lenstra '83, Kannan '87, Frank and Tardos '87]

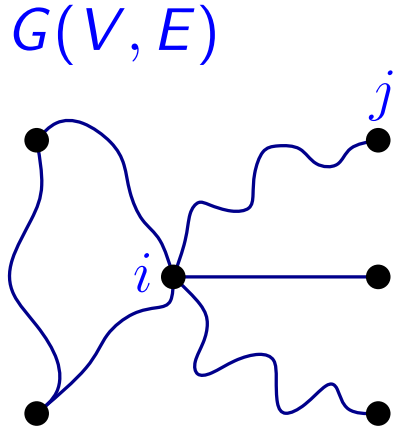
Vertex Cover



ILP...

ILP:

$$\begin{aligned} \text{min} \quad & \sum_{i \in V} x_i \\ \text{subject to} \quad & x_i + x_j \geq 1 \quad \forall ij \in E \\ & x_i \in \{0, 1\} \quad \forall i \in V \end{aligned}$$



ILP...and relaxation

LP:

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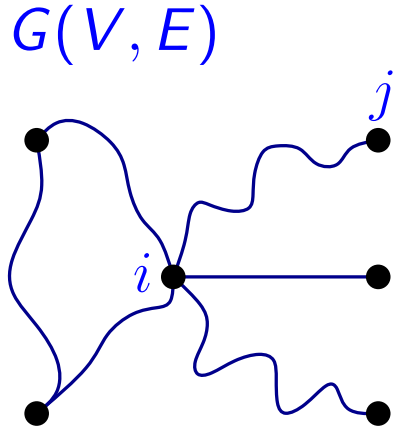

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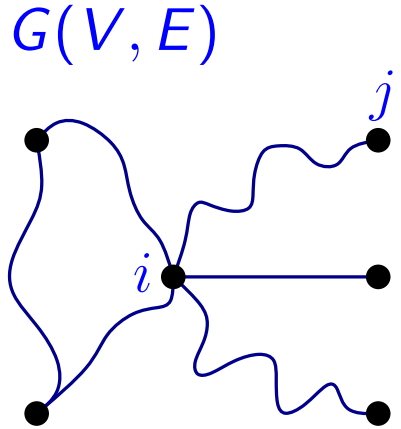

Main idea: Solve the LP, round the fractional solution to an integral solution that is feasible for ILP and whose cost is not 'much' more than the cost of LP solution

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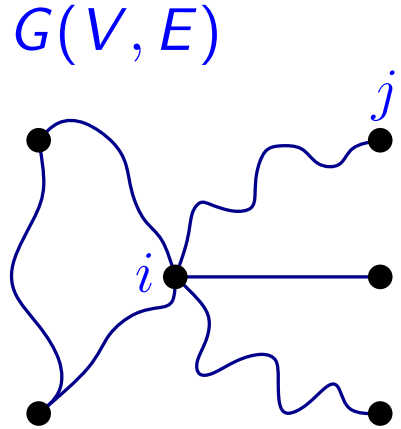

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Here: Let $\{x_i\}$ be an optimal solution for LP

Take $C = \{i \in V \mid x_i \geq 1/2\}$

Note that C is a cover with

$$|C| \leq \sum_{i \in V} 2x_i = 2OPT(LP) \leq 2OPT(ILP) \leq 2minVC$$

i.e., a 2-approximation

A combinatorial explanation for VC LP

(...or what is really happening)

$G(V, E)$

Theorem[Nemhauser and Trotter '75]:

1. $\exists$ opt. sol. $\{x_i\}$ for LP that is half-integral, i.e., $x_i \in \{0, 1/2, 1\}$ (and poly-time computable).

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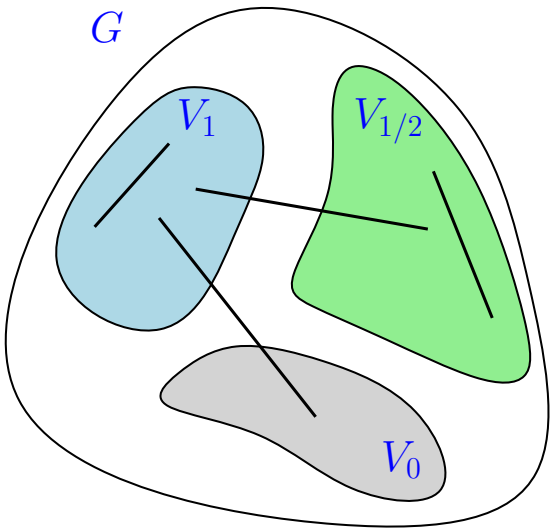

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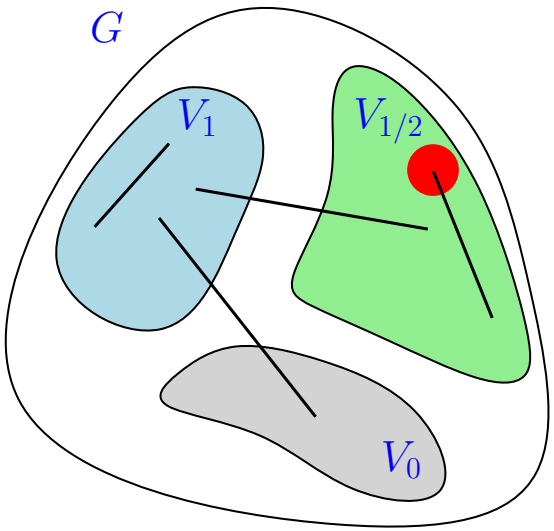

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(i) $\exists$ min.VC of G that consists of V_1 and of a min.VC of $G[V_{1/2}]$ i.e., $minVC(G[V_{1/2}]) + |V_1| = minVC(G)$

(ii) $minVC(G[V_{1/2}]) \geq |V_{1/2}|/2$

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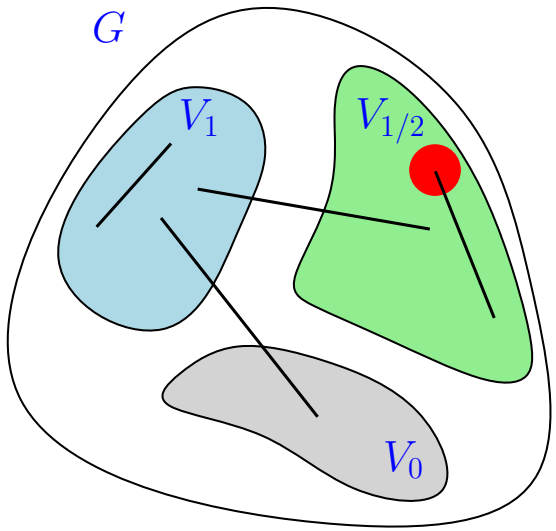

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In other words, forget V_0 and try to find a cover for $G[V_{1/2}]$

(Note: $V_1 \cup V_{1/2}$ gives a 2-approximation)

and...

...off to FPTness

Task: Given k, find a VC of size k or report that none exists

Simple: Focus on $G[V_{1/2}]$, reduce its size with the NT Thm!
[Chen, Kanj, Jia '01]

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Let $k' = k - |V_1|$ ('new' parameter with reduced value)

- $k' < 0 \implies$ NO (since $minVC \geq |V_1|$)
- $k' = 0 \implies$
  - NO, if $G[V_{1/2}]$ has at least one edge
  - YES, otherwise (V_1 is a cover)
- $k' > 0$ and $|V_{1/2}| > 2k' \implies$ NO (since $minVC(G[V_{1/2}]) > k'$)

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- **other:** 'new' instance (kernel) $(G[V_{1/2}], k')$ with $|V_{1/2}| \leq 2k'$ s.t.

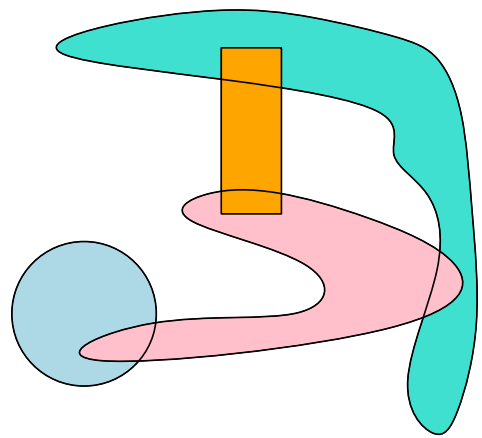
  G has a VC of size $k \iff G[V_{1/2}]$ has a VC of size k'

  Apply brute-force or an fpt-algorithm for Vertex Cover on $(G[V_{1/2}], k')$

More applications...

- LP-based Apx: > 100 papers (rand. rounding, primal-dual,...)
- $(2 - \frac{\log\log n}{2\log n})$-apx for Min. VC [Yehuda and Even '85]
- PTAS for Generalized VC (ext. of NT Thm.)
  [Yehuda and Hermelin and Rawitz '10]
- Kernel for Bounded Degree Deletion (ext. of NT Thm.)
  [Fellows, Guo, Moser, Niedermeier '09]
- Above Guarantee Vertex Cover
  (param.$=k - |maxMaching|$)(NT Thm. + preprocess. + branching)
  [Lokshtanov, Narayanaswamy, Raman, Ramanujan, Saurabh '11]

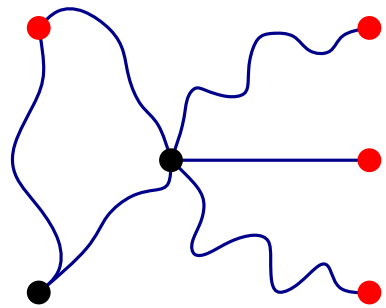
Independent set of (pseudo-) disks



Independent set in general

Maximum Independent Set

Given a graph $G(V, E)$, find a maximum-size independent set, i.e., a set $V' \subseteq V$ such that no two vertices in V' are connected by an edge.

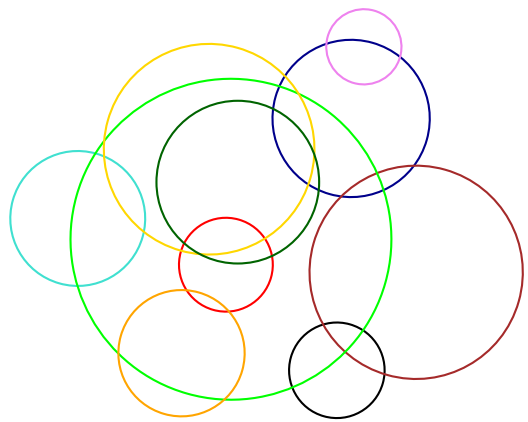


- no $O(n^{\epsilon-1})$-approximation algorithm, for any $\epsilon > 0$ (unless P=NP) [Håstad '99]
- W[1]-complete [Downey and Fellows '95]
- EPTAS for planar graphs: $O(2^{O(1/\epsilon)}n^2)$ [Baker '94]
- FPT (stand. param.) for planar graphs: $O(2^{O(\sqrt{k})}n)$ [Fomin and Thilikos '03], [Alber, Fellows, Niedermeier '04]

Independent set of geometric objects - something in between

Problem

Given a set $F = \{f_1, \ldots, f_n\}$ of n objects in $\mathbb{R}^2$ find a maximum-size subset of F such that no two objects intersect.



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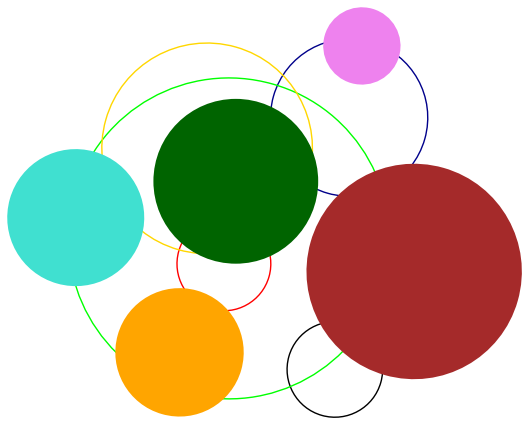

Reduces to Maximum Independent Set on the intersection graph of F

Independent set of geometric objects - something in between

Known:

- NP-hard (axis-parallel segments, unit disks, unit squares...)
- PTAS for disks, squares [Chan '03], [Erlebach, Jansen, Seidel '05]
- PTAS for pseudo-disks [Chan and Har-Peled '12]

- $O(\log \log n)$-approximation for axis-parallel rectangles [Chalermsook and Chuzhoy '03]
- $O(\sqrt{OPT})$-approximation for segments [Agarwal and Mustafa '06]

- W[1]-hard for unit disks, unit squares, unit segments with arbitrary directions [Marx '05], [Marx '06]
- FPT for λ-precision unit disks, segments with fixed directions [Alber and Fiala'04], [Marx '06], [Kára and Kratochvíl '06]

and many others...

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$\longrightarrow$ **O(1)-approximation for pseudo-disks** (weighted as well) [Chan and Har-Peled '12]

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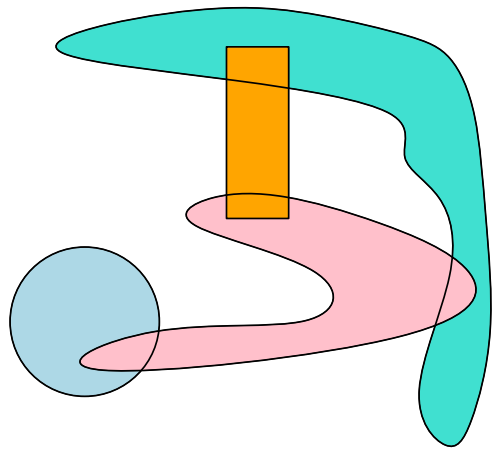
Pseudo-disks

A set of simply connected objects (regions) in $\mathbb{R}^2$, each bounded by a simple closed (Jordan) curve, is a collection of **pseudo-disks** if the boundaries of every two objects intersect at most twice.

pseudo-disks:



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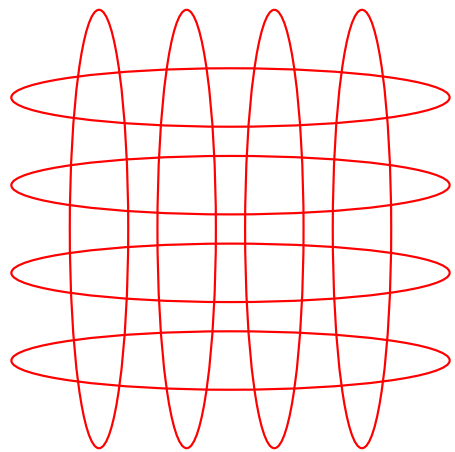
Not pseudo-disks:



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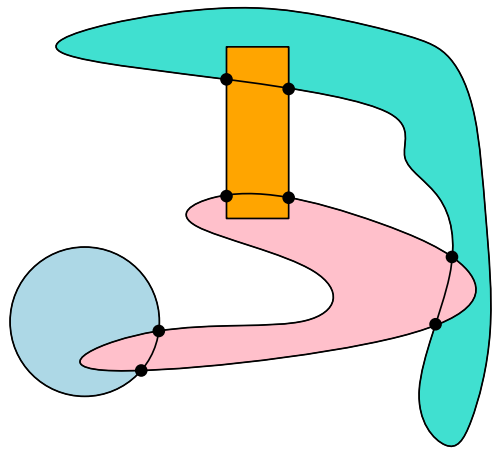


Linear union complexity:
The complexity of the (boundary of the) union of a set of k pseudo-disks is $O(k)$. In particular, there are at most $6k$ vertices on the boundary. [Kedem, Livne, Pach, Sharir '86]

(The following hold for any set with linear union complexity)

ILP and LP relaxation

$F = \{f_1, \ldots, f_n\}$: n objects in $\mathbb{R}^2$ (no full containment)

$\mathcal{A}(F)$: arrangement induced F, $\mathcal{V}(F)$: vertices of $\mathcal{A}(F)$

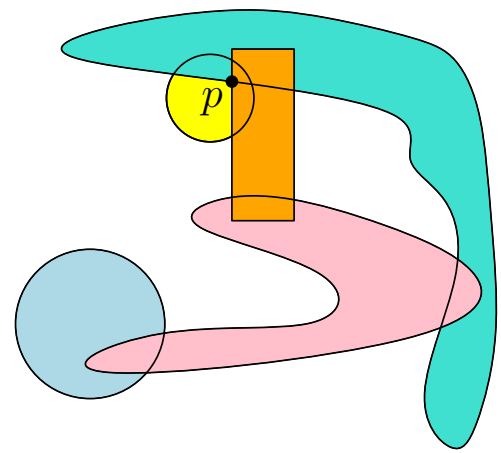
$x_i \leftarrow f_i$

LP:

$$\max \sum_{i \in [n]} x_i$$

$$\text{s. t.} \sum_{f_i \ni p} x_i \leq 1 \quad \forall \text{ vertex } p \in \mathcal{V}(F)$$

$$0 \leq x_i \leq 1 \quad \forall i \in [n]$$



Approach

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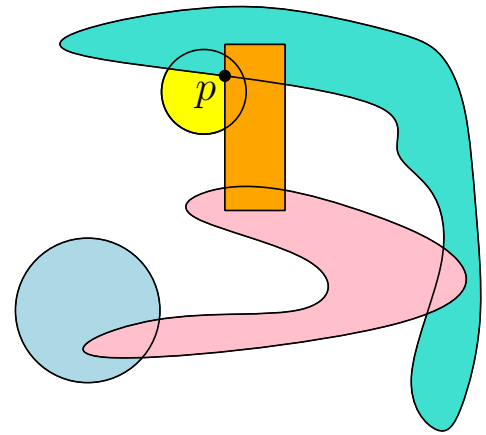

0. $R := \emptyset$.
1. Solve the LP: let $\{x_i\}$ be an optimal solution.
2. Randomly put each object f_i into R with probability x_i.

$\rightarrow$ **Show:** For the intersection graph $G(R, K)$ induced by R,
$E[|K|] = O(E[|R|])$

Approach

LP:

$$\max \sum_{i \in [n]} x_i$$

$$\text{s. t.} \sum_{f_i \ni p} x_i \leq 1 \quad \forall \text{ vertex } p \in \mathcal{V}(F)$$

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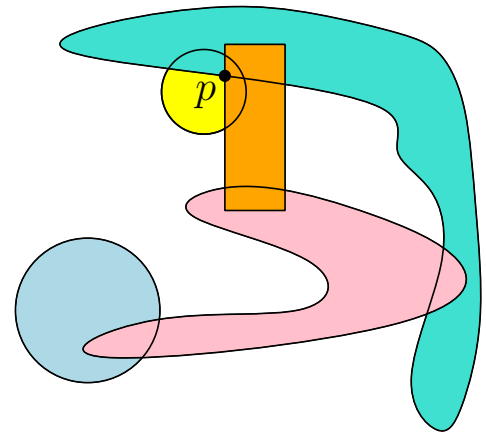

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$\rightarrow$ **Show:** For the intersection graph $G(R, K)$ induced by R,
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3. **Turán's Theorem:**
   Any graph with n vertices and average degree $\bar{d}$ has an independent set of size $\geq n/(\bar{d}+1)$.
   (e.g., use min. degree greedy algorithm)

Explanation...

$F = \{f_1, \ldots, f_n\}$: objects in $\mathbb{R}^2$ (and $R \subseteq F$: $|\mathcal{U}(R)| \leq \rho|R|$)

$\{x_i\}$: opt. sol. for LP, $G(R, K)$: induced intersection graph

Note: – $E[|R|] = \sum_{i \in [n]} x_i = OPT(LP) \geq maxIS$

– $E[|K|] = \sum_{i<j:\, f_i \cap f_j \neq \emptyset} x_i x_j$

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Lemma: $\sum_{i<j:\, f_i \cap f_j \neq \emptyset} x_i x_j \leq 4\rho E[|R|]$

Thus: $E[|K|] \leq 4\rho E[|R|]$

...and using Turán's Thm. on R we get an ind. set $I \subset R$ with

$$E[|I|] \geq E\left[\frac{|R|}{2K/|R| + 1}\right] \geq \cdots \geq \frac{E[|R|]}{(2E[|K|]/E[|R|]) + 1} \geq \frac{maxIS}{8\rho + 1}$$

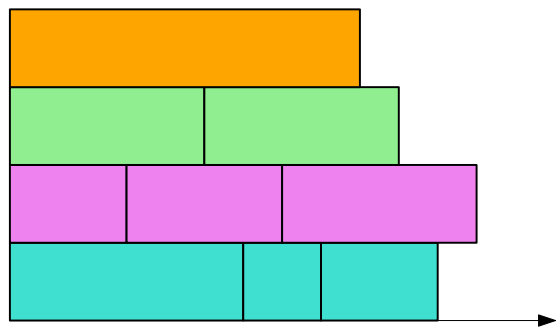
More on...

The $O(1)$-apx algorithm can be

- generalized to the weighted case
- derandomized

[Chan and Har-Peled '12]

Scheduling on identical machines



Scheduling on identical machines

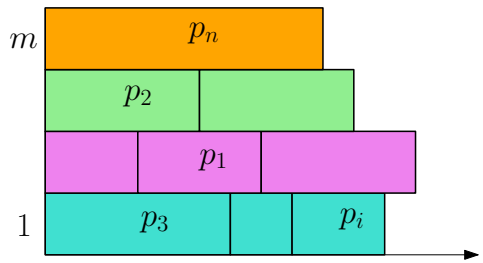
Minimum Makespan Scheduling

Given : n jobs, processing times $p_1, \ldots, p_n$, $p_j \in \mathbb{N}$
m identical machines (that run in parallel)

Compute : non-preemtive schedule that assigns each job to some machine such that the total processing time is minimized, i.e.,

a partition $S_1, \ldots, S_m$ of $[n]$ that minimizes

$$\max_{1 \le i \le m} \sum_{j \in S_i} p_j$$



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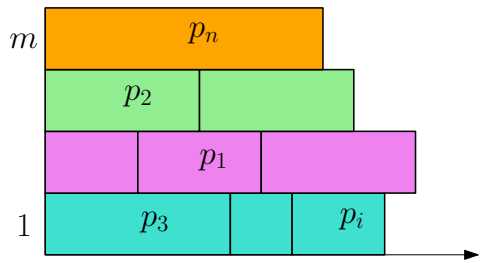

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- $4/3$-approximation [Graham '69]
- FPT (stand. param.) [Alon, Azar, Woeginger, Yadid '98]
- EPTAS [Alon, Azar, Woeginger, Yadid '98]: uses the fpt-algorithm!

Fixed-parameter tractbility via ILP

Makespan Scheduling:

Input : $m, p_1, \ldots, p_n, k \in \mathbb{N}$
Parameter : k
Question : Is there a schedule with makespan $\leq k$?

$n = f(k)$

ILP-F:

Input : $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$
Parameter : n (# variables)
Question : Is there $x \in \mathbb{Z}^n$ such that $Ax \leq b$?

Idea: Reduce Makespan Scheduling to ILP-F with $n = f(k)$ variables and use the fact that ILP-F is fixed-parameter tractable w.r.t n

1. Characterize assignments of cost $\leq k$

Instance: $m, p_1, \ldots, p_n, k \in \mathbb{N}$
Question: Is there a schedule of cost $\leq k$?

Observe:

- $p_i \leq k$, $\forall\, i \in [n]$ (otherwise answer NO)
- assignment A for a single machine:

  $\bar{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$, with $0 \leq a_i \leq k$, $\forall\, i \in [k]$

  i.e., exactly a_i jobs of cost i are assigned to the machine
  and $cost(\bar{a}) = \sum_{i \in [k]} a_i \cdot i$
- there are at most $(k+1)^k$ assignments of cost $\leq k$

2. Model schedules as ILP

$A = \{\text{assigment } \bar{\alpha} \text{ with } cost(\bar{\alpha}) \leq k\}$, $|A| \leq (k+1)^k$

Note: A schedule can be defined by the number of machines that follow an assignment $\bar{\alpha} \in A$

- variable $x_{\bar{a}} = \#$ machines following assignment $\bar{a}$, $\forall\, \bar{a} \in A$
- $b_j = \#$ jobs $i \in [n]$ with processing time $p_i = j$, $\forall\, j \in [k]$

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ILP:

$$\sum_{\bar{a} \in A} x_{\bar{a}} = m$$

(one assignment per machine)

$$\sum_{\bar{a}=(a_1,\ldots,a_k) \in A} a_j \cdot x_{\bar{a}} = b_j, \quad \forall j \in [k]$$

(right # jobs with processing time j is scheduled)

$$x_{\bar{a}} \in \mathbb{N}, \quad \forall \bar{a} \in A$$

(If ILP is feasible, a schedule can be easily found, e.g., by testing all possible choices for every variable)

EPTAS

Fact[Graham '66]: Greedy algorithm (*list scheduling*) gives a 2-approximation: *assign the next job to the first available machine.*

$L = \max\left(\{p_i \mid i \in [n]\} \cup \left\{\frac{1}{m}\sum_{i=1}^{n} p_i\right\}\right)$, we have $L \leq OPT$,

and the algorithm computes a feasible schedule of cost $\leq 2L$.

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Overview of EPTAS:

1. Given $k \in \mathbb{N}$, compute a schedule for the 'big' jobs ($p_i \geq L/k$) only, with cost that approximates OPT (whole instance) to a factor of $(1 + 1/k)$.

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   This reduces to (the standard parameterization) Makespan Scheduling for a scaled version of the restricted instance.

2. Schedule the 'small' ($p_i < L/k$) jobs so as not to increase the cost of by too much: Greedy will do.

More on...

Scheduling on unrelated machines
(different process. times on each machine):

- 2-apx. algorithm based on LP-relaxation
- no α-approximation algorithm exists for $\alpha < 3/2$

[Lenstra, Shmoys, Tardos '90]

Other applications of ILP to fixed-parameter tractability

- Fixed-parameter algorithms for Closest String and related problems [Gramm, Niedermeier, Rossmanith '03]
- Graph layout problems parameterized by (the size of the minimum) vertex cover
  [Fellows, Lokshtanov, Misra, Rosamond, Saurabh '08]

It's time for discussion!

Appendix

Parameterized complexity

Parameterized complexity theory provides a framework (hierarchy of classes) for showing fixed-parameter intractability

- the first level is the class W[1] (FPT$\subseteq$W[1])
- hardness is established via fpt-reductions
- a W[1]-hard problem is not in FPT, unless FPT=W[1]
- FPT=W[1] $\implies$ n-variable 3-SAT can be solved in $2^{o(n)}$ time (not believed to be true)

- Clique is W[1]-complete [Downey and Fellows '95]
- Independent Set is W[1]-complete [Downey and Fellows '95]

Scheduling big jobs...

Instance I: $m, p_1, \ldots, p_n$ with $p_1 \geq p_2 \geq \ldots \geq p_n$

$\bar{S} = (S_1, \ldots, S_m)$ an opt. schedule for I, $OPT(I) = \max_{1 \leq j \leq m} \sum_{i \in S_j} p_i$

Big jobs: jobs $i \in [n]$ with $p_i \geq L/k$

Assume: jobs $1, 2, \ldots, n'$ are big, for some $n' \in [n]$

New instance I': $m, p'_1, \ldots, p'_{n'}$ with $p'_i = \lfloor (k^2/L) \cdot p_i \rfloor$

Properties: $p'_i \geq k$ and $p_i \leq (1 + 1/k) \cdot (L/k^2) \cdot p'_i$

Lemma: $OPT(I') \leq 2k^2$.

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Properties: $p'_i \geq k$ and $p_i \leq (1 + 1/k) \cdot (L/k^2) \cdot p'_i$

Lemma: $OPT(I') \leq 2k^2$.

Compute an optimal schedule $\bar{S}' = (S'_1, \ldots, S'_m)$ for I' with $O(\log k)$ applications of the fpt-algorithm for Makespan Scheduling.

...scheduling big jobs

Schedule big jobs $1, \ldots, n'$ (processing times $p_i \geq L/k$) according to the oprimal schedule $\bar{S}'$ for the new instance I'

Properties of p'_i

$+$

for an opt. solution $\bar{S}$ for the whole instance I:

$S_1 \cap [n'], \ldots, S_m \cap [n']$ is a feasible solution for I' $\quad \Longrightarrow$

$$\max_{1 \leq j \leq m} \sum_{i \in S'_j} p_i \leq \cdots \leq \cdots\cdots\cdots\cdots$$
$$\leq (1 + 1/k) \cdot \max_{1 \leq j \leq m} \sum_{i \in S_j \cap [n']} p_i$$

...scheduling big jobs

Schedule big jobs $1, \ldots, n'$ (processing times $p_i \geq L/k$) according to the oprimal schedule $\bar{S}'$ for the new instance I'

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$$\leq (1 + 1/k) \cdot \max_{1 \leq j \leq m} \sum_{i \in S_j \cap [n']} p_i$$

$$= (1 + 1/k) \cdot OPT(I)$$

- $n' = n$: DONE

...scheduling big jobs and small jobs

Schedule big jobs $1, \ldots, n'$ (processing times $p_i \geq L/k$) according to the oprimal schedule $\bar{S}'$ for the new instance I'

Properties of p'_i

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$$= (1 + 1/k) \cdot OPT(I)$$

- $n' = n$: DONE
- $n' < n$: schedule the small ($p_i < L/k$) jobs $n' + 1, \ldots, n$ greedily, obtain a schedule $\bar{S}^*$

$$\max_{1 \leq j \leq m} \sum_{i \in S^*_j} p_i \leq (1 + 1/k) \cdot OPT(I)$$