A General Framework for Reasoning About Change

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Abstract The capability to represent and use concepts like time and events in computer science are essential to solve a wide class of problems characterized by the notion of change. Real-time, databases and multimedia are just a few of several areas which needs good tools to deal with time. Another area where this concepts are essential is artificial intelligence because an agent must be able to reason about a dynamic environment.

In this work a formalism is proposed which allows the representation and use of several features that had been recognized as useful in the attempts to solve such class of problems. A general framework based in a many-sorted logic is proposed centering our attention in issues such as the representation of time, actions, properties, events and causality. The proposal is compared with related work from the temporal logic and artificial intelligence areas. This work complements and enhances previously related efforts on formalizing temporal concepts with the same purpose.

Keywords Knowledge Representation, Temporal Logic, Time, Events, Actions, Causality

§1 Introduction

The capability to represent and use concepts like time and events are essential in computer science to solve a wide class of problems characterized by the notion of change. Real-time, databases and multimedia are just a few of several areas which needs good tools to deal with time. Temporal reasoning is also a core area of artificial intelligence (AI) as it is necessary for an intelligent agent to know how to behave in a dynamic world. It is for this reason that the artificial intelligence community is becoming increasingly interested in formalizing temporal reasoning since two decades ago. [10]

One of the most influential proposals in AI has been that of Allen's interval-based temporal logic. [2] [5] The goal of this article is to provide an alternative general framework based on a a many-sorted temporal logic that allows temporal reasoning and gives a cohesive account of previous work [16] [2] [17] [18] [38] We consider syntax and semantics with a detailed specification of each of the involved sorts, as well as other complementary notions like individualization and causality.

Because of the diversity of aspects involved when temporal phenomena is considered, one is faced with making choices. It is worth mentioning that there is no strong philosophical commitment in our definitions and assumptions. The decisions are mostly based on computational reasons. This does not mean we do not pay attention to the philosophical literature. Instead we try to get a balance between what is known from the philosophical literature and the computational implications in considering them. It is also important to remark we will not be involved in this article with all problems arising from the initially stated goal. For example, we will not consider the frame problem or non-monotonic aspects of reasoning, whose relation with this proposal will be considered in a future article.

This work is organized as follows. The language and the specification of sorts, including some remarks on causality, are provided in section 2. Inference rules, semantics and the problem of individuation arising on a reified logic are considered in section 3. Examples on how to use the proposed framework are given in sections 2.7 and 4. A comparison between our proposal and similar frameworks is done in section 5, showing how much some well-known proposals of the literature accomodates in this work. Finally, conclusions and future work are given in section 6.

§2 Syntax of the Temporal Language

Following we will give the specification of a many-sorted temporal language called $\mathcal{L}^{\mathbb{T}}$. This will provide us with tools to naturally consider reification over time, properties, events and actions. These have been considered in the literature of the area as key concepts on modelling a rational agent living in a dynamic world. The reason for choosing reification is that it brings us some advantages from the point of view of knowledge representation and use. As Allen has pointed out, [3] we need reification in order to efficiently represent information in the knowledge base when we deal with incomplete knowledge. Also it is useful to efficiently handle the problem to distinguish two individuals, e.g. if we need to know if two events are different or not. In addition, we think many sorted logics offer a clear framework to specify all these classes of individuals neatly separated and this separation also could lead to a more efficient computational treatment. There are extensive studies of many-sorted logics with functions and equality, giving syntax, semantics, proof theory and metatheoretical properties [17]. Then we do not need to reinvent it and instead we shall focus on the extension of such a general framework to do it suitable for reasoning with temporal concepts connecting past work and personal work to get a more precisely defined proposal. We start with a reminder of a sorted signature to be used in $\mathcal{L}^{\mathbb{T}}$.

Definition 2.1
The *alfabet* of $\mathcal{L}^{\mathbb{T}}$ has the following elements:

1. We assume a countable set S of *sorts* such that $S \supseteq \{\mathcal{B}, \mathcal{E}_x, \mathcal{E}_v, \mathcal{P}, \mathcal{A}, \mathcal{W}\}$ where: sort $\mathcal{B}$ is for boolean values, i.e., *true* and *false*, sort $\mathcal{E}_x$ is for explicit temporal references and $\mathcal{E}_v$ is for event-based temporal references, $\mathcal{P}$ is the sort for properties and $\mathcal{A}$ for actions. $\mathcal{W}$ will contain the remaining individuals existing in the application domain. For each sort s we consider: a) two *quantifiers*: $\forall_s$ and $\exists_s$ b) an *equality symbol* $\doteq_s$ c) a countable, possibly infinite, set V_s of *variables* d) a countable and non-empty set CS_s of *constants*.
2. A set of *boolean connectives* : $\wedge$, $\vee$, $\rightarrow$, $\neg$ and a set of auxiliary symbols: $), ($ and "$,$".
3. A countable, possibly empty, set FS of *function symbols*, $f_0, f_1, \ldots$, together with a *rank function* $r : FS \rightarrow S^+ \times S$, assigning to each function symbol f a pair $r(f) = (u, s)$, called *rank*, where u represents the arity of f and s the sort of the result after applying f to its arguments. The set of boolean connectives $\wedge$, $\vee$ and $\rightarrow$ can be then considered as functions with rank $(\mathcal{B}.\mathcal{B}, \mathcal{B})$ and the connective $\neg$ with rank $(\mathcal{B}, \mathcal{B})$.

4. A countable and non-empty set PS of *predicate symbols*, $P_0, P_1, \ldots$, together with a rank function $r : PS \to S^\star \times \mathcal{B}$, assigning each predicate symbol P a pair $r(P) = (u, s)$, called the rank, where u represents the sorts for the arguments of P and s the sort of the result after applying P to its arguments. When the arity is zero, P is a propositional letter. Equality, $\doteq_s$, is considered a predicate with rank $(s.s, \mathcal{B})$.

Next we define the well-formed formulas of the language $\mathcal{L}^{\mathbb{T}}$ [8] [9], in a BNF-style for brevity's sake:

Definition 2.2
Let $s_i, s_j, \ldots, s_m, s_n$ be sort names in S, the set of *well-formed formulas of* $\mathcal{L}^{\mathbb{T}}$, wff, is defined as follows:

term_{s_m}	::=	$\text{variable}_{s_m} \mid \text{constant}_{s_m} \mid \text{function\_name}_{s_m}\ (\ \mathit{term\_list}\ )$
$\mathit{term\_list}$	::=	$\mathit{term}_{s_i} \mid \mathit{term}_{s_j}\ ,\ \mathit{term\_list}$
$\mathit{atomic\_formula}$	::=	$\text{predicate\_name}\ (\ \mathit{term\_list}\ )$
wff	::=	$\mathit{atomic\_formula} \mid (\ \mathit{term}_{s_n}\ \doteq_{s_n}\ \mathit{term}_{s_n}\ ) \mid (\neg\ \mathit{wff}\ ) \mid$
		$(\ \mathit{wff}\ \to\ \mathit{wff}\ ) \mid (\ \mathit{wff}\ \wedge\ \mathit{wff}\ ) \mid (\ \mathit{wff}\ \vee\ \mathit{wff}\ ) \mid$
		$((\exists_{s_n}\ \text{variable}_{s_n}\ )\ \mathit{wff}\ ) \mid ((\forall_{s_n}\ \text{variable}_{s_n}\ )\ \mathit{wff}\ )$

NOTATION: Symbols of each sort, as $\doteq_s$ and $\forall_s$, are used only with symbols of the same sort. When it is clear from the context we will omit the subscript to specify the intended sort. Also we take the convention of numbering just the axioms of the theory. Nested negations are ruled out, i.e. all formulas of the form $\neg\neg F$ will be considered as equivalent to F. We will use $\forall_s x, y...$ or $\exists_s x, y...$ instead of $\forall_s x \forall_s y...$ and $\exists_s x \exists_s y....$ We will also use $x{>}y, x{<}y{<}z, x \sqsubseteq y \sqsubseteq z$ instead $y{<}x, x{<}y \wedge y{<}z, x \sqsubseteq y \wedge y \sqsubseteq z$ respectively. We will proceed analogously when using "<" and symbols from other sorts.

Examples of valid temporal constants are: 7-8-1991 and 3600. These could be used as a date and the amount of seconds in an hour respectively. Examples of functions are: a) *leap-year(A)* that maps a year into the constants *true* and *false* in the expected way and *seconds_year(A)* that maps a year in the set of natural numbers according to the amount of seconds that the considered year has. Some examples of atemporal terms are: choral$(magnificat)$ and author$(magnificat, jsbach)$, representing respectively a kind of musical composition and the authorship property of a person over a composition. If we consider $\mathcal{E}_x, \mathcal{P}, \mathcal{E}_v, \mathcal{A}$ and $\mathcal{W}$ as names of sorts the following are examples of well formed formulas in $\mathcal{L}^{\mathbb{T}}$:

$\exists_{\mathcal{T}} i_1, i_2\ \mathit{Precedes}(i_1, i_2)$

$\exists_{\mathcal{A}} a \exists_{\mathcal{E}} e \exists_{\mathcal{P}} p \exists_{\mathcal{T}} i(\text{Do}_{at}(a, i) \wedge \text{Occurs}_{at}(e, i+1) \to \text{Holds}_{at}(p, i+2))$

$\forall_{\mathcal{T}} i_1, i_2(\neg\ \mathit{Precedes}(i_1, i_2) \wedge \neg\ \mathit{Precedes}(i_2, i_1) \to\ \mathit{Simultaneous}(i_1, i_2))$

$\text{Occurs}_{on}(born(jsbach), 21/3/1685)$

In the following section we will give an axiomatization of the sorts $\mathcal{E}_x, \mathcal{P}, \mathcal{E}_v, \mathcal{A}$ of interest for our purposes of representing an agent with temporal reasoning capabilities. The sort $\mathcal{W}$ will remain without specification because it depends on the particular domain to be modelled. Unlike other proposals in the literature [2] we do not consider the notion of process as an essential one. We do this under the hypothesis that they could be constructed from events and states. [18] [38]

2.1 The sort $\mathcal{E}_x$

We will provide in this section an axiomatization of the sort of explicit temporal references, $\mathcal{E}_x$. Here we use "explicit temporal references" and "explicit time" as a short way to refer to numeric, absolute, calendric kind of temporal references. On the other hand by "implicit time" we mean non-numeric, relative, event-based, temporal references. We could consider two kinds of references which in turn define subsorts of $\mathcal{E}_x$. We could start its definition considering one of these subsorts, that defined by "instants". Later we shall consider the other subsort, defined by "intervals". Here the word "subsort" has a weak meaning in the sense that intervals are built from instants instead as a completely independent notion. This step will be left as a further refinement to this proposal. By an instant we mean the shortest temporal measure with respect to the granularity assumed on the system being modelled. An instant must not be considered here as durationless, instead it is the name of the unit of measure assumed in the system (which in some articles is called *chronos*). This is a point-based conception of time over which we shall later construct an interval-based structure. The subsort $\mathcal{T}$ is formalized in the structure $INS : \langle \mathcal{T}, < \rangle$ where $\mathcal{T}$ is a set of points of time termed "instants" and $<: \mathcal{T} \times \mathcal{T}$ is an order relation. The following axioms are valid in $\mathcal{T}$ (Notation: we usually shall denote members of $\mathcal{T}$ by i and its subscripts):

$$\forall i_1 \ \neg(i_1 < i_1) \tag{1}$$

$$\forall i_1, i_2, i_3 (i_1 < i_2 \wedge i_2 < i_3 \rightarrow i_1 < i_3) \tag{2}$$

$$\forall i_1 \ \exists i_2 (i_2 < i_1) \tag{3}$$

$$\forall i_1 \ \exists i_2 (i_1 < i_2) \tag{4}$$

$$\forall i_1, i_2 (i_1 < i_2 \vee i_2 < i_1 \vee i_1 \doteq i_2) \tag{5}$$

$$\forall i_1, i_2 (i_1 < i_2 \rightarrow \exists i_3 (i_1 < i_3 \wedge \neg \exists i_4 i_1 < i_4 < i_3)) \tag{6}$$

$$\forall i_1, i_2 (i_1 < i_2 \rightarrow \exists i_3 (i_3 < i_2 \wedge \neg \exists i_4 i_3 < i_4 < i_2)) \tag{7}$$

which characterise an irreflexive, transitive (hence asymmetric), nonending and discrete line of time. This excuses us from considering some characteristic problems of other structures but absent in discrete frameworks such as the intermingling problem [20] and the specification of the moment of change in a property. [40] It is also important to notice we are not assuming the structure as ismorphic to $\mathbb{Z}$, which allows us to give an entirely first-order axiomatization. We now define a notion of interval over INS as a subsort inside $\mathcal{E}_x$, which will be represented by means of $\mathcal{I}$.

The usual method to build intervals in similar frameworks is to consider them as a set of instants. Here we do not choose this way because of problems that arise in considering the ocurrence of events associated to intervals in relation with its non-homogeneity property. That is to say, usually it is considered that if an event occurs in an interval conceived as a set of instants it also occurs in the set of instants that defines it. This conflicts with the non-homogeneity hypothesis over events. Since we are assuming events as non-homogeneous it is more adequate to associate an interval with a pair of instants considering it as a unit. Nothwithstanding, the points delimiting the interval allow us to do a kind of instant-based and constraint-based reasoning that has been proved very useful in temporal reasoning. [27]

Definition 2.3
We will call an *interval* each member of the set $\mathcal{I} = \{[i_1, i_2] \in \mathcal{T} \times \mathcal{T} \mid i_1 < i_2\}$. We shall also consider the partial function *int* mapping elements of $\mathcal{T} \times \mathcal{T}$ on elements of $\mathcal{I}$: $int(i_1, i_2) =_{def} [i_1, i_2]$ if $i_1 < i_2$.

NOTATION: We shall change the usual parenthesis associated with ordered pairs for brackets to align them to the usual appeareance in the temporal reasoning literature. We shall also usually denote intervals by I and its subscripts.

As it could be noticed, we are discarding "punctual intervals", i.e., intervals of the form $[i, i]$. This is so because being simultaneously an instant and also an interval both its meaning and their set of properties would be ambiguous.

Definition 2.4
We will consider the total functions $begin, end : \mathcal{I} \rightarrow \mathcal{T}$ that give us for each interval their beginning and ending points respectively: $begin([i_1, i_2]) =_{def} i_1$ and $end([i_1, i_2]) =_{def} i_2$.

Now we could consider a structure $INT : \langle \mathcal{I}, \lessdot, \sqsubseteq \rangle$ where $\mathcal{I}$ is a set of intervals and $\sqsubseteq$ (previous than), $\lessdot$ (subinterval) relations, with $\lessdot, \sqsubseteq \subseteq \mathcal{I} \times \mathcal{I}$, defined as follows:

$$I_1 \lessdot I_2 =_{def} \{(I_1, I_2) \in \mathcal{I} \times \mathcal{I} \mid end(I_1) < begin(I_2)\}$$

$$I_1 \sqsubseteq I_2 =_{def} \{(I_1, I_2) \in \mathcal{I} \times \mathcal{I} \mid begin(I_2) \leq begin(I_1), end(I_1) \leq end(I_2)\}$$

Also we will use the following definition (also obtainable from $\sqsubseteq$):

$$I_1 \cdot\!\!=\!\!\cdot\; I_2 =_{def} begin(I_1) \doteq begin(I_2) \wedge end(I_1) \doteq end(I_2)$$

Instants and intervals were considered as abstract entities so far. Lets consider how they would look in an everyday scenario.

Example 2.1
In order to represent dates we could take the extend western notation as a triple $mm/dd/yyyy$ meaning respectively *month/day/year*. For example, 01/20/2000 means *january the 20th, 2000*. If temporal granularity or chronos of the system is fixed at days, then $01/20/2000 \in \mathcal{T}$ and the order relation $<$ is the algorithm allowing us to say if a date is earlier than another or not. If temporal granularity is seted to minutes then $I = 01/20/2000 \in \mathcal{I}$ and $begin(I) = 01/20/2000, 00 : 01am$ and $end(I) = 01/20/2000, 11 : 59pm$. The way to denote dates and clock time was left to be decided at implementation time. The system was defined free of a particular way to denote this entities. More on the way to chose interval limits in section 5.5.

Because of the temporal entities introduced, now we can define a set of well-known relations in the literature as those between intervals of Hamblin, later adopted by Allen, [1] and those between points and intervals: [27]

Lemma 2.1
Interval relations BEFORE, MEETS, OVERLAPS, BEGINS, DURING, FINISHES, EQUALS, their inverses and the following relations between points and intervals: *precedes*, *start*, *divides*, *ends*, *follows* can be defined in $\mathcal{T}$.

Proof As was shown by Allen and Hayes, [6] BEFORE, OVERLAPS, BEGINS, DURING, FINISHES and EQUALS could be defined from MEETS. However, we will rewrite them according to our terminology to show how simple is in our framework and to allow future citations of them.

$$
\begin{array}{lll}
BEFORE(I_1, I_2) & =_{def} & end(I_1) < begin(I_2) \\
MEETS(I_1, I_2) & =_{def} & end(I_1) \doteq begin(I_2) \\
OVERLAPS(I_1, I_2) & =_{def} & begin(I_1) < begin(I_2) < end(I_1) < end(I_2) \\
BEGINS(I_1, I_2) & =_{def} & begin(I_1) \doteq begin(I_2) \wedge end(I_1) < end(I_2) \\
DURING(I_1, I_2) & =_{def} & begin(I_2) < begin(I_1) \wedge end(I_1) < end(I_2) \\
FINISHES(I_1, I_2) & =_{def} & begin(I_2) < begin(I_1) \wedge end(I_1) \doteq end(I_2) \\
EQUALS(I_1, I_2) & =_{def} & I_1 \cdot\!=\!\cdot I_2 \\
Precedes(i, I) & =_{def} & i < begin(I) \\
Start(i, I) & =_{def} & i \doteq begin(I) \\
Divides(i, I) & =_{def} & begin(I) < i < end(I) \\
Ends(i, I) & =_{def} & i \doteq end(I) \\
Follows(i, I) & =_{def} & end(I) < i
\end{array}
$$

The reader can check this facts directly from the previously cited works [1) fig. 1] [27) table 4.1] after appropriate notation changes. ■

We could also get similar theorems to axiom 5 (we skip the proof for the sake of brevity):

$$\forall I_1, I_2(BEFORE(I_1, I_2) \vee MEETS(I_1, I_2) \vee OVERLAPS(I_1, I_2) \vee BEGINS(I_1, I_2) \vee$$
$$DURING(I_1, I_2) \vee FINISHES(I_1, I_2) \vee EQUALS(I_1, I_2))$$

$$\forall i, I(Precedes(i, I) \vee Start(i, I) \vee Divides(i, I) \vee Ends(i, I) \vee Follows(i, I))$$

We can identify some general properties [38)] about the structures $\mathcal{T}$ and $\mathcal{I}$. Both satisfies SYMMETRY, *i.e.*, seeing to the future and the past is not different, CONNECTION, *i.e.*, all pairs of elements are related, and HOMOGENEITY, *i.e.*, all elements have the same properties.

2.2 The sort $\mathcal{E}_v$

We attempt here a reconstruction of reasoning about change without explicit time. For this purpose we will use a framework similar to that adopted for the sort $\mathcal{E}_x$, this time splitting $\mathcal{E}_v$ into two subsorts $\mathcal{N}$ and $\mathcal{D}$ for non-durative (punctual) and durative events respectively. Here the word subsort has the same weak meaning that in the previous section in relation to instants and intervals. In a later chapter we will present our proposal as an improvement of that given by Kamp [23)] and a simplification of that of Thomason. [32)]

An *event structure* $PUN : (\mathcal{N}, <_{\mathcal{E}})$ is considered where $\mathcal{N}$ is a set of punctual events and $<_{\mathcal{E}} \subseteq \mathcal{N} \times \mathcal{N}$ is a binary order relation. The following axioms hold in the structure:

$$\forall e \neg(e <_{\mathcal{E}} e) \tag{8}$$

$$\forall e, e', e''(e <_{\mathcal{E}} e' \wedge e' <_{\mathcal{E}} e'' \rightarrow e <_{\mathcal{E}} e'') \tag{9}$$

$$\forall e \exists e'(e' <_{\mathcal{E}} e) \tag{10}$$

$$\forall e \exists e'(e <_{\mathcal{E}} e') \tag{11}$$

We consider a relation of *punctual simultaneity* represented by $S_p \subseteq \mathcal{N} \times \mathcal{N}$ where:

$$e_1 S_p e_2 =_{def} \neg(e_1 <_{\mathcal{E}} e_2 \vee e_2 <_{\mathcal{E}} e_1)$$

Because linearity does not follow from the previous axioms then we state the following:

$$\forall e, e', e''(eS_pe' \wedge e'S_pe'' \rightarrow eS_pe'') \tag{12}$$

Lemma 2.2
S_p defines an equivalence relation over $\mathcal{N}$.

Proof Reflexivity is given because $e_1S_pe_1 =_{def} \neg(e_1 <_{\mathcal{E}} e_1 \vee e_1 <_{\mathcal{E}} e_1) = \neg(e_1 <_{\mathcal{E}} e_1)$ that is obtained by axiom 8. For symmetry we must look if $e_2S_pe_1$ when $e_1S_pe_2$. By definition of S_p, we could consider instead if whenever is true $\neg(e_1 <_{\mathcal{E}} e_2 \vee e_2 <_{\mathcal{E}} e_1)$ then $\neg(e_2 <_{\mathcal{E}} e_1 \vee e_1 <_{\mathcal{E}} e_2)$ is true. But this is clearly the case because it is just a matter of order in the disjunctive clauses. Transitivity is assured by axiom 12. ■

Analogously to the sort $\mathcal{T}$ we will suppose:

$$\forall e_1, e_2(e_1 <_{\mathcal{E}} e_2 \vee e_2 <_{\mathcal{E}} e_1 \vee e_1S_pe_2) \tag{13}$$

$$\forall e_1, e_2(e_1 <_{\mathcal{E}} e_2 \rightarrow \exists e_3(e_1 <_{\mathcal{E}} e_3 \wedge \neg\exists e_4(e_1 <_{\mathcal{E}} e_4 <_{\mathcal{E}} e_3))) \tag{14}$$

$$\forall e_1, e_2(e_1 <_{\mathcal{E}} e_2 \rightarrow \exists e_3(e_3 <_{\mathcal{E}} e_2 \wedge \neg\exists e_4(e_3 <_{\mathcal{E}} e_4 <_{\mathcal{E}} e_2))) \tag{15}$$

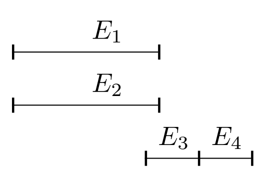
The *event structure* $DUR : (\mathcal{D}, Begin_{\mathcal{E}}, End_{\mathcal{E}})$ comprises a set $\mathcal{D}$ of durative events. We assume each durative event has naturally associated the events of "starting to occur" and "ceasing to occur". Then analogously to the case of intervals we consider two functions, $Begin_{\mathcal{E}}, End_{\mathcal{E}} : \mathcal{D} \rightarrow \mathcal{N}$, by means of which we could obtain the punctual events denoting respectively the beginning and ending of a given durative event. For each event $E \in \mathcal{D}$, we impose the restriction $Begin_{\mathcal{E}}(E) <_{\mathcal{E}} End_{\mathcal{E}}(E)$. We easily get the relations *durative simultaneity* represented by S_d, *overlapping events* represented by O_d, and *abutting* represented by A_d. All these relations are defined as "*Relation*" $\subseteq \mathcal{D} \times \mathcal{D}$:

$$ES_dE' =_{def} Begin_{\mathcal{E}}(E) = Begin_{\mathcal{E}}(E') \wedge End_{\mathcal{E}}(E) = End_{\mathcal{E}}(E')$$

$$EO_dE' =_{def} Begin_{\mathcal{E}}(E) <_{\mathcal{E}} Begin_{\mathcal{E}}(E') <_{\mathcal{E}} End_{\mathcal{E}}(E)$$

$$EA_dE' =_{def} End_{\mathcal{E}}(E) = Begin_{\mathcal{E}}(E')$$

Example 2.2
Let us suppose the following events,



then we have $E_1S_dE_2$, $E_2O_dE_3$ and $E_3A_dE_4$.

2.3 Bridging the gap between $\mathcal{E}_x$ and $\mathcal{E}_v$

It is useful to provide means of conexion between sorts $\mathcal{E}_x$ and $\mathcal{E}_v$ because sometimes, paradigmatically in AI, we do not have complete knowledge about the world being formalized. In this section we consider ways to connect knowledge from two sorts increasing the chances to infer implicit knowledge. Lets consider as a way of example the simplest combination. If we are told that two punctual events e_1 and e_2 occurred but not precisely when we could infer their order relation by connecting them with their times of occurrence, t_1 and t_2. If $t_1 < t_2$ then we can infer that e_1 is previous to e_2. The goal of this article is to draw several ways of connecting the diferent temporal references considered in $\mathcal{E}_x$ and $\mathcal{E}_v$. We start using the recently defined notion of event to define temporal notions associated without explicit time references.

Definition 2.5
Let $PUN : (\mathcal{N}, <_{\mathcal{E}})$ be a structure of punctual events, as the simultaneity relation defines an equivalence relation over $\mathcal{N}$, we could identify an "instant" with each simultaneity class so defined over $\mathcal{N}$. Also we will consider a function $e\_instant : \mathcal{N} \rightarrow \mathcal{T}$ that returns a name of an instant associated with the simultaneity class that a given punctual event belongs to.

Definition 2.6
Let $DUR : (\mathcal{D}, Begin_{\mathcal{E}}, End_{\mathcal{E}})$ be a structure of durative events. A set $\{E_1, E_2, \ldots, E_k\}$ of elements from $\mathcal{D}$ will be termed a *chain of events*, abreviated as ξ, in the following cases:

1. every set of events related by S_d is a chain of events
2. if $\{E_1, \ldots, E_n\}$ is a chain of events and E_{n+1} is a set of simultaneous non-atomic events such that for $E \in E_{n+1}$ either $E_n O_d E$ or $E_n A_d E$, then $E_1, \ldots, E_n, E_{n+1}$ is a chain of events.

Example 2.3
The set of events described in example 2.2 allow us to define the following chains: $\{E_1\}$, $\{E_1, E_3\}$, $\{E_1, E_3, E_4\}$, $\{E_2\}$, $\{E_2, E_3\}$, $\{E_2, E_3, E_4\}$, $\{E_1, E_2\}$, $\{E_1, E_2, E_3\}$, $\{E_1, E_2, E_3, E_4\}$.

It is interesting to observe we have now several options to define order relations over $\mathcal{E}_v$. We start considering an order relation defined over punctual events.

Definition 2.7
Let *e_instant* be the function defined before over punctual events, then we define the order relation $<'$ as:

$$e\_instant(e_1) \; <' \; e\_instant(e_2) =_{def} e_1 <_{\mathcal{E}} e_2$$

Lemma 2.3
Let $INS' = \langle \mathcal{T}', <' \rangle$ be a structure of instants defined from a set of punctual events, then $<'$ defines a strict linear order over $\mathcal{T}'$.

Proof we must prove that given two instants i_1 and i_2, they verify:

1. $i_1 <' i_2 \rightarrow \neg(i_2 <' i_1)$
2. $i_1 <' i_2 \wedge i_2 <' i_3 \rightarrow i_1 <' i_3$
3. $i_1 \neq i_2 \rightarrow i_1 <' i_2 \oplus i_2 <' i_1$ (where $\oplus$ means disjunctive union)

As $\mathcal{T}'$ is a set of punctual events and $<'$ is an order relation based on $<_{\mathcal{E}}$ the proof is trivial from axioms 9 plus simultaneity and asymmetry that are obtained from axioms 8 and 9. All we need to do is to translate in a suitable way assertions about instants onto the involved events axioms through the linking definitions 2.5 and 2.7. ■

We proceed now to do similar steps regarding durative events. We start defining some auxiliary functions which are used to define new order relations. Later we state some results as done for the punctual events case.

Definition 2.8
If $\mathcal{D}$ is a set of events, ξ is a chain of events from $\mathcal{D}$ as in definition 2.6, and $2^{\mathcal{D}}$ the power set of $\mathcal{D}$, then the function $First : 2^{\mathcal{D}} \rightarrow \mathcal{D}$ returns one of the events starting before the rest of the set (discarding simultaneity). Analogously, $Last : 2^{\mathcal{D}} \rightarrow \mathcal{D}$ returns one of the events ending after the rest of the set (again discarding simultaneity).

$$First(\xi) =_{def} \begin{cases} E & \text{if} \quad E \in \xi \text{ and there is no } E' \in \xi(Begin_{\mathcal{E}}(E') <_{\mathcal{E}} Begin_{\mathcal{E}}(E)) \\ undefined & \quad\text{otherwise} \end{cases}$$

$$Last(\xi) =_{def} \begin{cases} E & \text{if} \quad E \in \xi \text{ and there is no } E' \in \xi(End_{\mathcal{E}}(E) <_{\mathcal{E}} End_{\mathcal{E}}(E')) \\ undefined & \quad\text{otherwise} \end{cases}$$

Definition 2.9
Let $DUR = (\mathcal{D}, Begin_{\mathcal{E}}, End_{\mathcal{E}})$ be a structure of events. We can consider event-based intervals as the temporal extent associated with every non-empty subset of $\mathcal{D}$ that form a chain of events. The function $e\_interval : 2^{\mathcal{D}} \rightarrow \mathcal{I}$ return the interval associated to a set of durative events. Then, $e\_interval$ is a function assigning names of intervals to sets of events.

Definition 2.10
Let $e\_instant$ and $e\_interval$ be the functions defined before, and ξ_1, ξ_2 two chain of events. Then we will consider the following orders:

$$e\_interval(\xi_1)B'e\_interval(\xi_2) =_{def} e\_instant(Begin_{\mathcal{E}}(First(\xi_1))) <' e\_instant(Begin_{\mathcal{E}}(First(\xi_2)))$$
$$e\_interval(\xi_1)\mathbf{B}'e\_interval(\xi_2) =_{def} e\_instant(End_{\mathcal{E}}(Last(\xi_1))) <' e\_instant(Begin_{\mathcal{E}}(First(\xi_2)))$$

Lemma 2.4
Given $INS_1 = \langle \mathcal{T}', B' \rangle$ and $INS_2 = \langle \mathcal{T}', \mathbf{B}' \rangle$ structures of instants defined from a set of punctual events. Neither B' nor $\mathbf{B}'$ defines a strict linear order over $\mathcal{T}'$.

Proof It is enough to see that in both cases the third axiom fails. ■

Now we can go through one of the main goals of this section, that is to establish a relation between the sorts $\mathcal{E}_x$ and $\mathcal{E}_v$. Most of the previous definitions were defined with the purpose to fulfil this goal acting as a bridge between explicit time-based and event-based knowledge. In principle, no concrete time ("clock time") is assumed to be associated with events. Typically, the knowledge or data base will have a set of punctual and durative events designators representing an implicit time together with some order constraints. There will also be references to instants and intervals. Sometimes we could connect all this information referring to an event taking advantage of inferences on one way to represent temporal

concepts and the other relating both views of the information. For example, the notions of temporal precedence over instants and intervals could be reconstructed in terms of those defined for events.
We could consider the functions $p\_events : \mathcal{D} \to \mathcal{N} \times \mathcal{N}$ such that $p\_events(e) = (Begin_{\mathcal{E}}(E), End_{\mathcal{E}}(E))$ and $de\_instants : \mathcal{N} \times \mathcal{N} \to \mathcal{T} \times \mathcal{T}$ such that $de\_instants(e_1, e_2) = (e\_instant(e_1), e\_instant(e_2))$. From the previous definitions some relations could be established between elements of $\mathcal{E}_x$ and $\mathcal{E}_v$ as indicated in figure 1.

$$
\begin{tikzcd}
\mathcal{N} \times \mathcal{N} \arrow[r, "de\_instants"] & \mathcal{T} \times \mathcal{T} \arrow[d, "int"] \\
\mathcal{D} \arrow[u, "p\_events"] \arrow[r, dotted, "e\_interval"'] & \mathcal{I}
\end{tikzcd}
$$

Fig. 1 Translating durative events to intervals.

We do not have a function like *e_interval* in figure 1, but we are able to obtain the same result through the composition of *p_events*, *de_instants* and *int*, in these order. We could also transform the information the other way around. If we consider functions $points : \mathcal{I} \to \mathcal{T} \times \mathcal{T}$ where $points(I) =_{def} (begin(I), end(I))$ that is to say the inverse of *int*, and $d\_event : \mathcal{N} \to \mathcal{D}$ by means of which we obtain the durative event associated to the punctual events denoting the beginning and the end of its ocurrence. Using them we could transform knowledge over punctual events into instant based knowledge as indicated in figure 2.

$$
\begin{tikzcd}
\mathcal{N} \times \mathcal{N} \arrow[r, dotted, "de\_instants"] \arrow[d, "d\_event"'] & \mathcal{T} \times \mathcal{T} \\
\mathcal{D} \arrow[r, "e\_interval"'] & \mathcal{I} \arrow[u, "points"']
\end{tikzcd}
$$

Fig. 2 Translating punctual events to instants.

In this case the imposibility to apply the function *de_instants* can be replaced by the composition of *d_event*, *e_interval* and *points*.

Example 2.4
Let E be a durative event: “the stone fell from my hand to the floor”, which ocurrence starts in the punctual event e_1 of “release the stone” and concludes in the punctual event e_2 by which the stone touches the floor. Let us suppose that associated interval to this event is $[17:05:00, 17:05:02]$ on the form *hour* : *minutes* : *seconds*. By using the functions associated to events we could recover the initial events by means of: $Begin_{\mathcal{E}}(E) = e_1$ and $End_{\mathcal{E}}(E) = e_2$. By using *e_instant* we obtain $e\_instant(e_1) = 17:05:00$

and $e\_instant(e_2) = 17:05:02$. After applying $int(17:05:00, 17:05:02) = [17:05:00, 17:05:02]$ we get the interval associated to the event and apply to it operations associated with the sort $\mathcal{I}$. We could also go from punctual events to their associated instants.

Then we have ways to get information from all kinds considered via simple functions. What makes the difference between both concepts, explicit time and events, is that while events could be used to denote time they are also associated with change. Not just as a way to detect when change occurs but as a change producer. We now make explicit considerations about the way in which events relate with explicit time and individuals of other sorts. Let $\mathrm{Occurs}_{at}(e, i)$ and $\mathrm{Occurs}_{on}(E, I)$ denote the occurrence of a punctual event e in the moment i and the occurrence of a durative event E on interval I respectively. They are related in the following sense:

$$\forall_{\mathcal{D}}\ E\ \forall_{\mathcal{I}}\ I\ \forall_{\mathcal{T}}\ i\ (\mathrm{Occurs}_{on}(E, I) \wedge \mathrm{In}(i, I) \rightarrow \neg\mathrm{Occurs}_{at}(E, i))$$

where $\mathrm{In}(i, I) =_{def} \mathrm{Start}(i, I) \vee \mathrm{Divides}(i, I) \vee \mathrm{Finishes}(i, I)$. To relate event-based and "explicit time"-based reasoning we could consider:

$$\forall_{\mathcal{N}}\ e_1, e_2\ \forall_{\mathcal{T}}\ i_1, i_2\ (\mathrm{Occurs}_{at}(e_1, i_1) \wedge \mathrm{Occurs}_{at}(e_2, i_2) \wedge e_1 <_{\mathcal{E}} e_2 \rightarrow i_1 < i_2) \tag{16}$$

$$\forall_{\mathcal{N}}\ e_1, e_2\ \forall_{\mathcal{T}}\ i_1, i_2\ (\mathrm{Occurs}_{at}(e_1, i_1) \wedge \mathrm{Occurs}_{at}(e_2, i_2) \wedge i_1 < i_2 \rightarrow e_1 <_{\mathcal{E}} e_2) \tag{17}$$

Similarly, we can derive the following axioms from our knowledge of two durative events defining intervals of time that are known to have no part in common.

$$\forall_{\mathcal{D}}\ E_1, E_2\ \forall_{\mathcal{I}}\ I_1, I_2\ (\mathrm{Occurs}_{on}(E_1, I_1) \wedge \mathrm{Occurs}_{on}(E_2, I_2) \wedge E_1 \mathbf{B}' E_2 \rightarrow I_1 \lessdot I_2)$$

$$\forall_{\mathcal{D}}\ E_1, E_2\ \forall_{\mathcal{I}}\ I_1, I_2\ (\mathrm{Occurs}_{on}(E_1, I_1) \wedge \mathrm{Occurs}_{on}(E_2, I_2) \wedge I_1 \lessdot I_2 \rightarrow E_1 \mathbf{B}' E_2)$$

This can be done in the following way: If we know $\mathrm{Occurs}_{on}(E_1, I_1) \wedge \mathrm{Occurs}_{on}(E_2, I_2) \wedge E_1 \mathbf{B}' E_2$ then by definition of $\mathbf{B}'$ and durative events we have $Begin_{\mathcal{E}}(E_1) <_{\mathcal{E}} End_{\mathcal{E}}(E_1) <_{\mathcal{E}} Begin_{\mathcal{E}}(E_2) <_{\mathcal{E}} End_{\mathcal{E}}(E_2)$. Throughout the functions *de_instants* and *e_interval* we can associate to E_1 and E_2 two intervals I_1 and I_2 such that $I_1 \lessdot I_2$. With the second axiom we proceed analogously using the function *d_event* and then using functions $Begin_{\mathcal{E}}$ and $End_{\mathcal{E}}$ to get the elements where to apply transitivity. We get the conclusion by definition of $\mathbf{B}'$. It is worth to notice that it is not possible to derive 17 from 16 and our previous definitions and axioms. This is because we cannot use, as we did in sort $\mathcal{P}$, the homogeneity property to derive what happens in the interval from what we know about their subintervals. On the contrary, from the non-homogeneity of events we are forced to state 17 as an axiom. Then, up to axiom 17 we could get a purely instant-based theory in the way interval reasoning could be represented in terms of them. After this point interval occurrences must be imposed as axioms. Finally, we consider weak negation over durative events in the following sense:

$$\forall_{\mathcal{D}}\ E\ \forall_{\mathcal{I}}\ I\ (\neg\mathrm{Occurs}_{on}(E, I) \leftrightarrow \exists_{\mathcal{T}}\ i\ (\mathrm{In}(i, I) \wedge \neg\mathrm{Occurs}_{at}(E, i))) \tag{18}$$

2.4 The sort $\mathcal{P}$

For the representation of properties we will consider predicates introduced by Galton: [18] $\mathrm{Holds}_{at}(p, i)$ and $\mathrm{Holds}_{on}(p, I)$ denoting that p is a property that is true in the moment i or interval I respectively. Actually, we just use p as a simplification because instead of what is expected, we will have something like $A(x)$ to assert that x possesses the property A. We can see that $\text{Holds}_{on} \subseteq \mathcal{P} \times \mathcal{I}$ and $\text{Holds}_{at} \subseteq \mathcal{P} \times \mathcal{T}$ are related in the following sense:

$$\text{Holds}_{on}(p, I) =_{def} \forall_{\mathcal{T}}\ i\ (\text{In}(i, I) \rightarrow \text{Holds}_{at}(p, i))$$

From the previous definition we get the following theorems about homogeneity of properties over an interval:

$$\forall_{\mathcal{P}}\ p \forall_{\mathcal{T}}\ i\ \forall_{\mathcal{I}}\ I\ (\text{Holds}_{on}(p, I) \wedge \text{In}(i, I) \rightarrow \text{Holds}_{at}(p, i)$$

$$\forall_{\mathcal{P}}\ p \forall_{\mathcal{I}}\ I, I'\ (\text{Holds}_{on}(p, I) \wedge I' \sqsubseteq I) \rightarrow \text{Holds}_{on}(p, I'))$$

We consider "weak negation" of properties over intervals that could be obtained directly from the negation of the previous definition:

$$\neg\text{Holds}_{on}(p, I) =_{def} \exists_{\mathcal{T}} i(\text{In}(i, I) \wedge \neg\text{Holds}_{at}(p, i))$$

In what follows we will use a relation $\text{Changes}(e, p) : \mathcal{E}_v \times \mathcal{P}$ denoting that e is an event, either punctual or durative, which everytime it occurs provokes the change of the property p. The relation between the change of a property and a previous occurrence of an event is stated through:

$$\begin{aligned}
&\forall_{\mathcal{P}}\ p\ \forall_{\mathcal{T}}\ i, i'\ ((\text{Holds}_{at}(p, i) \wedge \neg\text{Holds}_{at}(p, i')) \vee (\neg\text{Holds}_{at}(p, i) \wedge \text{Holds}_{at}(p, i')) \wedge i < i' \rightarrow \\
&\qquad \exists_{\mathcal{E}v}\ e\ (\text{Changes}(e, p) \wedge \\
&\qquad [\exists_{\mathcal{T}} i''(\text{Occurs}_{at}(e, i'') \wedge i'' < i') \vee \exists_{\mathcal{I}} I(\text{Occurs}_{on}(e, I) \wedge\ \mathit{begin}(I) < i')]) \qquad (19)
\end{aligned}$$

It must be observed that in spite of the homogeneity of properties the set of combinations could not be extended to consider a property changing its truth value from an instant to an interval, viceversa or changing between consecutive intervals. This is because in spite of *weak* negation a property ceasing to hold at an interval do not implies not holding in the beginning and ending points. To allow this kind of inferences *strong* negation, i.e. a property does not holds in an interval if it does not at every instant of that interval, must be previously added to the system.

2.5 The sort $\mathcal{A}$

In some contexts it is difficult to differentiate one action from the event that it causes, e.g. John's flipping a switch. Then the reader could think if it is really necessary to have another sort for actions since its consideration leads sometimes to possibly artificial differentiations between them. In this article we will follow the hypothesis that this feature is convenient to the option of not being allowed to distinguish them when so it is needed. We will consider that every action is performed by an agent:

$$\forall_{\mathcal{A}}\ a\ (\text{Action}(a) \rightarrow \exists_{\mathcal{W}}\ g\ \text{Agent}(a, g)) \qquad (20)$$

More axioms including actions but in relation with events and causality are considered in another section below.

2.6 Actions, Events and Causality

This section is devoted to a general formalization of the notion of causality in relation to temporality. This section must not be considered as a full account of causality, instead it was conceived as a way to connect notions introduced in previous sections. As a positive side effect we give details about fundamental concepts to be considered in a later discusion about Allen's proposal for causality. The axioms impose general constraints on predicates related to causality. They must be suplemented with other axioms bringing knowledge about special aspects of causality between specific events and actions for particular scenarios. We concede that much of the following treatment would find a better account in a non-monotonic framework but, as argued before, such considerations are beyond the scope of the present work.

One issue to bear in mind is that in this proposal actions are considered different objects from events altough this is not always the case all areas of computer science. Here we take this view in spite of pragmatic advantages, as it allow us to state more clearly the different parts which play a role in the description of change. We will consider first action causality and as a first hypothesis we will suppose that an event cannot be previous to the action that produces it. It is worth mentioning that this hypothesis has many supporters in the philosophical literature but not unanimity. For example, there are proponents of the argument that all causes are simultaneous with their effects [39] while Davidson's [16)pp. 158] representation of causality limits itself to causes that strictly precede effects. We consider then two situations. One option is that the beginning points could be simultaneous like perceiving the colour of an object when we look at it or producing sounds while pulling the stick over the strings of an instrument. Also they could be overlapping as when somebody pushes an object during a period until it collides with some object. Another example is the promulgation of a law that is made in a particular date but whose effects could begin later.

We will also assume that an event produced by an action can finish before the end of the action that produces it. One scenario in which this could happen is when an action produces an instantaneous event. For example, my action of spilling water out of a glass provokes the event of starting to spread the water or the event of the first contact of water with the floor. Another situation of interest arises when the effect of the action ceases before it is expected. For example, when somebody passes a bow over the strings of an instrument and after some time the sound ceases because a string breaks. Regarding Allen's assumptions, the only surviving hypothesis is that the action cannot start later than the event it causes.

Let us suppose we use the predicate Acause as a relation $\text{Acause} \subseteq \mathcal{A} \times \mathcal{E}_v$ denoting that an action causes an event occurrence. Also we consider Do_{at} and Do_{on} with actions in a similar spirit than Occurs_{at} and Occurs_{on} for events denoting instantaneous actions, like snapping the fingers or blinking the eyes, and durative actions, such as raising the arm, respectively. We could resume the previous ideas as follows:

$$\begin{aligned}&\forall_{\mathcal{N}}\ e\ \forall_{\mathcal{T}}\ i_e\ \exists_{\mathcal{A}}\ a\ (\text{Occurs}_{at}(e, i_e) \land\ \text{Acause}(a, e) \rightarrow \\ &\qquad [\exists_{\mathcal{T}}\ i_a\ (\text{Do}_{at}(a, i_a) \land i_a \le i_e) \lor \exists_{\mathcal{I}}\ I_a\ (\text{Do}_{on}(a, I_a) \land\ \mathit{begin}(I_a) \le i_e)]) \end{aligned} \tag{21}$$

$$\begin{aligned}&\forall_{\mathcal{D}}\ E\ \forall_{\mathcal{I}}\ I_e\ \exists_{\mathcal{A}}\ a\ (\text{Occurs}_{on}(E, I_e) \land\ \text{Acause}(a, E) \rightarrow \\ &\qquad [\exists_{\mathcal{T}}\ i_a\ (\text{Do}_{at}(a, i_a) \land\ i_a \le \mathit{begin}(I_e)) \lor \exists_{\mathcal{I}}\ I_a\ (\text{Do}_{on}(a, I_a) \land\ (\mathit{begin}(I_a) \le \mathit{begin}(I_e)))]) \end{aligned} \tag{22}$$

As with action causation we introduce a predicate, Ecause as a relation $\text{Ecause} \subseteq \mathcal{E}_v \times \mathcal{E}_v$, denoting that there exists a correlation between two event occurrences. Actually, this is a simplification of the problem because it could be argued that this could be considered as a relation $\text{Ecause} \subseteq \mathcal{P} \times \mathcal{E}_v \times \mathcal{E}_v$. The reason that could be given is that properties are needed to allow the causing event to occur. Similar, but weaker, arguments could be given to include the properties that change as part of the effect. We opted, to simplify this point, the assumption that there are rules in the knowledge base to link properties with events in an appropriate way to specify the dependence of events with its associated properties. Similarly to Acause we consider the following axioms on event causation:

$$\forall_{\mathcal{N}}\ e\ \forall_{\mathcal{T}}\ i\ \exists_{\mathcal{E}_{\mathcal{V}}}\ e'\ (\text{Occurs}_{at}(e,i) \wedge\ \text{Ecause}(e',e) \rightarrow$$
$$[\exists_{\mathcal{T}}\ i'\ (\text{Occurs}_{at}(e',i') \wedge i' \leq i) \vee \exists_{\mathcal{I}}\ I'\ (\text{Occurs}_{on}(e',I') \wedge\ \mathit{begin}(I') \leq i)]) \tag{23}$$
$$\forall_{\mathcal{D}}\ E\ \forall_{\mathcal{I}}\ I\ \exists_{\mathcal{E}_{\mathcal{V}}}\ e'\ (\text{Occurs}_{on}(E,I) \wedge\ \text{Ecause}(e',E) \rightarrow$$
$$[\exists_{\mathcal{T}}\ i'\ (\text{Occurs}_{at}(e',i') \wedge\ i' \leq \mathit{begin}(I)) \vee \exists_{\mathcal{I}}\ I'\ (\text{Occurs}_{on}(e',I') \wedge\ \mathit{begin}(I') \leq\ \mathit{begin}(I))]) \tag{24}$$

We will add more on the relation between causation and time after introducing some refinement to the Ecause relation. We consider also two types of event causations, *direct* and *indirect* causation. The purpose of that is to distinguish between two classes of causality with different properties. This is an attempt to distinguish the set of events $\{e_{i_1}, \ldots, e_{i_n}\}$ that are the direct cause for another event e from the set of events $\{e_{k_1}, \ldots, e_{k_m}\}$ that were the cause for $\{e_{i_1}, \ldots, e_{i_n}\}$. Then besides the relation Ecause we introduce the relations $DEcause, IEcause \subseteq \mathcal{E}_v \times \mathcal{E}_v$ denoting if an event e_1 is or not a direct cause of another event e_2. We can resume this by the following definitions:

$$\begin{array}{lll} \text{Ecause}(e_1,e_2) & =_{def} & \text{DEcause}(e_1,e_2) \vee \text{IEcause}(e_1,e_2) \\ \text{DEcause}(e_1,e_2) & =_{def} & \text{Ecause}(e_1,e_2) \wedge \neg\text{IEcause}(e_1,e_2) \end{array}$$

This distinction does not seem to be fundamental. However, it could help to discuss some differences about properties of causation as will be done later. Also it has some pragmatic advantages. If it is needed not to take in to account all the details in a reasoning it could be convenient to do skips with respect to the granularity of the knowledge available. For example, in a legal context it would be of interest to deduce something related to the reason of death of a person. This would be centered in direct physical causes or indirect reasons of the context on which it happened. In a related problem, there is a difference in the sentence for a person that kills another and those that help him to do it. The first would be naturallly described as who directly cause the direct cause of death and his accomplices as whom indirectly caused the direct casue of death. In another context, we could be interested in explaining the allergic reaction a man got by taking the wrong pill without reference to all the chemical and biological processes involved in his body. However, altough most of the time the cause is just unknown, we often do it regardless this is not the direct cause of the event but an indirect one. Then, in our theory we can distinguish when we do not know if the event is a direct or an indirect one by using ECause. If we know that it is an indirect causation or we want to use it to avoid a too much small-grain sized reasoning we can use IEcause. If we know precisely that it is a direct causation we have DEcause at our disposal. Each kind of causation has different properties. If there were some pair of events that depending on the context could be well related either by direct or by indirect causation they could be put as related by Ecause or give them different names to differentiate the direct causation from the indirect one.

To see that it is not true that indirect axioms satisfy anti-symmetry it is enough to think in some devices like a set of pending metallic balls, aligned in such a way that there is a physical contact between them. If somebody takes one ball on an end and releases it, the impact, e_1, of the released ball on the following one is transmitted by the others until the last is separated and in the inverse movement it plays the role of the first ball shocking, e_2, with the rest when it comes back in the inverse direction. The process is repeated and causes the first ball to collide in the same way as previously and so on. Indirectly, e_1 causes e_2 and e_2 causes e_1, then in some cases like these, indirect events are not anti-symmetric. In the same scenario, if we have that e_1 causes e_2 and e_2 causes e_1, by transitivity we have that e_1 indirectly causes itself then e_1 is also reflexive. Another scenarios that support this view is a pendulum clock that functions with weights, like old clocks of sixteen and seventeen centuries and billards games. We could also consider if DEcause is anty-symetric and it seems that this is not the case. Let us suppose a boy on an end of a seesaw. The event of the boy walking from one end of the seesaw to the other causes the table to go down. Also, under some assumptions, the seesaw table going down causes the boy to walk from one point to the other. Then each event causes the other and in some circumstances we do not have anti-simmetry even in direct event causation. Then we could just consider one axiom setting that transitivity implies indirect event causation and one axiom discarding anti-reflexivity just on direct event causation:

$$\forall_{\mathcal{E}_V}\ e, e', e''\ (\text{Ecause}(e, e') \wedge \text{Ecause}(e', e'') \leftrightarrow \text{IEcause}(e, e'')) \tag{25}$$

$$\forall_{\mathcal{E}_V}\ e, e'\ (\text{DEcause}(e, e') \rightarrow e \neq e') \tag{26}$$

It is interesting to note that from the previous axioms and definitions we cannot expect to have some axioms relating Direct and Indirect causality with time in the following way:

$$\text{DEcause}(e, e') \rightarrow [\forall_{\mathcal{I}}\ I(\text{Occurs}_{on}(e, I) \rightarrow \exists_{\mathcal{I}}\ I'(\text{Occurs}_{on}(e', I') \wedge\ \text{MEETS}(I, I')))]$$

$$\text{IEcause}(e, e') \rightarrow [\forall_{\mathcal{I}}\ I(\text{Occurs}_{on}(e, I) \rightarrow \exists_{\mathcal{I}}\ I'(\text{Occurs}_{on}(e', I') \wedge\ \text{BEFORE}(I, I')))]$$

Direct causation does not imply "being immediately followed by" and indirect causation does not imply "not being immediately followed by". We could think of the following scenarios to see that this is the case. For the first case it is enough to consider again the situation where the boy walks over the seesaw provoking its going down. This is a direct causation and both events, the boy walking from one ending to the other and the wood moving from one position to the other, are simultaneous. Also the promulgation of a law is a direct cause to make the law start to be applied after a period. Then, in this last case we have a temporal gap in the time between the causing event and the caused event. For indirect causation we could consider a stack of two blocks, to say T over B, and some event causing the block B to move, for example an explosion under B. The movement of B directly causes the movement of T. But the explosion under B indirectly caused the movement of T and they are simultaneous events. Furthermore, when we established that e_1 causing e_2 could be indirect if there was an event caused by e_1 that causes e_2 we are not ruling out other events in the middle and therefore, a temporal gap between e_1 and e_2. Then it seems all we can say is that the caused event cannot start before the causing event, as was asserted through a previous axiom.

2.7 Using $\mathcal{L}^{\mathbb{T}}$ as a Specification Language

Before completing the definition of this proposal we consider an illustration on how $\mathcal{L}^{\mathbb{T}}$ can be used to specify temporal requirements in different contexts.

Example 2.5
Real-time systems [22] [26] provide means to guarantee response times to various events occurring in these systems. The possibility to specify durations is a key issue in this area because it allows to state deadlines. In this way it is possible to establish that some steps in the behaviour of the system must be achieved without exceeding these critical times at the reisk of reaching an undesired situation. We can see below how we can use $\mathcal{L}^{\mathbb{T}}$ to specify some restrictions to the behaviour of a gas burner, a well-known problem of the area.

"*Gas must never leak for more than 4 time units in any period of at most 30 units*"
$length(J) =_{def} end(J) - begin(J)$ for any interval $J \in \mathcal{I}$
$\forall_{\mathcal{I}} I, L\ (\mathrm{Occurs}_{on}(leaks, L) \wedge DURING(L, I) \wedge length(I) \leq 30 \rightarrow length(L) \leq 4)$

Example 2.6
Synchronization is a key issue in multimedia systems [21] because it is essential to have a way to specify order in duration notions between different media items to be sure that the presentation follows an organized plan. Multimedia items can a) have a fixed duration or last as long as the user wants b) have a relative location regarding other events or being launched at a fixed time after the beginning of the presentation or even be shown at a special day and time of a year c) be shown once or in a repetitive way d) be conditioned by other events or totally independent. system-user interaction must also be considered because it can have a significant impact in the presentation even leading to replanning, e.g. when a user reverses a video or jumps some scheduled items in a presentation. We consider first some examples of synchronization specifications in order to show how they can be written using $\mathcal{L}^{\mathbb{T}}$.

*) "*If the video finishes or it is stopped, its associated sound file must be also stopped*".

$$\forall_{\mathcal{T}} t\ (\mathrm{Do}_{at}(\mathrm{user\_stop}(presentation.mpg), t) \vee \mathrm{Occurs}_{at}(\mathrm{end\_video}(presentation.mpg), t) \rightarrow$$
$$\mathrm{Holds}_{at}(\mathrm{ends\_display}(presentation.mpg), t+1))$$

*) "*Simultaneously with the beginning of the presentation starts to show a rotating logo for 30 seconds and play Vivaldi's Gloria. After that gradually blur the logo and lower the volume in 4 seconds*".

$$\forall_{\mathcal{T}} t\ (\mathrm{Occurs}_{at}(start, t) \rightarrow$$
$$\mathrm{Occurs}_{on}(play(GloriaVivaldi.wav), [t, t+30]) \wedge \mathrm{Occurs}_{on}(show(logo.dxf), [t, t+30]) \wedge$$
$$\mathrm{Occurs}_{on}(blur(logo.dxf), [t{+}31, t{+}35])) \wedge \mathrm{Occurs}_{on}(decreasevol(GloriaVivaldi.wav), [t+31, t+35]))$$

*) "*After the logo display was started show a video until the logo finishes (after 3 minutes, measured in seconds) or the user stops it*".

$$\forall_{\mathcal{T}} t\ (\mathrm{Occurs}_{on}(show(logo.dxf), L) \wedge (L = [t, t+30]) \rightarrow$$
$$\exists_{\mathcal{D}} E \exists_{\mathcal{I}} I\ (\mathrm{Name}(E, display\_video)\ \wedge \mathrm{Occurs}_{on}(\mathrm{display\_video}(presentation.mpg), I) \wedge$$
$$((\mathrm{OVERLAPS}(L, I) \wedge\ (length(I) = 180)) \vee (\exists_{\mathcal{N}} u\ (\mathrm{Nombre}(u, user\_push\_stop)\ \wedge End_{\mathcal{E}}(E) S_p u))$$

§3 Inference Rules and Semantics

We assume a set of inference rules that could be briefly described as a Gentzen system for many-sorted logics with equality.[17)12)] Gallier [17)] gives a detailed exposition of the metatheoretical properties and the proof procedure associated to a many-sorted logic with these inference rules.

We have defined several kinds of individuals then we are also concerned with the problem of deciding when two expressions of our logic refer to the same individual. We had considered at the end of the last section this problem with respect to equality. This provides the same general theory of equality for all sorts. Here we consider particularities associated with the individuation of members of each sort. The justification to pursuit such an individuation besides the equality inference rules previously stated is that some kind of sort asks for a more detailed way to specify when two individual references are supposed to be the same. This is of particular interest in artificial intelligence applications where knowledge about the world is supposed to be usually poor and we need to find other means to get implicit knowledge from previous explicit knowledge. In sorts $\mathcal{W}$, $\mathcal{P}$ and $\mathcal{E}_x$ we do not provide further ways to prove equality than those previously introduced through the inference rules. In sort $\mathcal{T}$, given its set of axioms, we could derive

$$\forall i_1, i_2 (\neg(i_1 < i_2) \wedge \neg(i_2 < i_1) \rightarrow i_1 \doteq i_2)$$

For the sort $\mathcal{I}$ we have defined

$$I_1 \cdot\!=\!\cdot\ I_2 =_{def} \text{begin}(I_1) \ \doteq\ \text{begin}(I_2) \ \wedge\ \text{end}(I_1) \ \doteq\ \text{end}(I_2)$$

In the sort $\mathcal{E}_v$ we consider that two events are distinguishable from their spatio-temporal location

$$\forall_{\mathcal{N}}\ e_1, e_2\ \forall_{\mathcal{T}}\ i\ \forall_{\mathcal{W}}\ l \quad (\text{Occurs}_{at}(e_1, i) \wedge \text{Occurs}_{at}(e_2, i) \wedge \text{At}(e_1, l) \wedge \text{At}(e_2, l) \leftrightarrow e_1 = e_2) \tag{27}$$

$$\forall_{\mathcal{D}}\ E_1, E_2\ \forall_{\mathcal{I}}\ I\ \forall_{\mathcal{W}}\ l \quad (\text{Occurs}_{on}(E_1, I) \wedge \text{Occurs}_{on}(E_2, I) \wedge \text{At}(E_1, l) \wedge \text{At}(E_2, l) \leftrightarrow E_1 = E_2) \tag{28}$$

It is worth mentioning that there is a justification in the previous axioms for a spatial sort where designate that something is a location. This is useful here because when we specify a location in the present framework we must choose an individual from $\mathcal{W}$ and in this sort there is a wide range of them. We will not explicitly define a special sort considering all individuals of $\mathcal{W}$ as a possible location for the ocurrence of an event. We consider that two actions in the sort $\mathcal{A}$ are distinguishable by spatio-temporal coordinates and the agent that performs them.

$$\forall_{\mathcal{A}}\ a_1, a_2\ \forall_{\mathcal{T}}\ i\ \forall_{\mathcal{W}}\ g, l \quad (\text{Do}_{at}(a_1, i) \wedge \text{Do}_{at}(a_2, i) \wedge \text{At}(a_1, l) \wedge \text{At}(a_2, l) \wedge \text{Agent}(a_1, g) \wedge \text{Agent}(a_2, g) \leftrightarrow a_1 = a_2) \tag{29}$$

$$\forall_{\mathcal{A}}\ a_1, a_2\ \forall_{\mathcal{T}}\ i\ \forall_{\mathcal{W}}\ g, l \quad (\text{Do}_{on}(a_1, i) \wedge \text{Do}_{on}(a_2, i) \wedge \text{At}(a_1, l) \wedge \text{At}(a_2, l) \wedge \text{Agent}(a_1, g) \wedge \text{Agent}(a_2, g) \leftrightarrow a_1 = a_2) \tag{30}$$

It could be considered a many-sorted algebra based semantics [17)] for $\mathcal{L}^{\mathbb{T}}$ as follows. The different sorts are carriers and each sort s_k has its own function mapping terms, possibly from diferent sorts, to terms in the sort s_k, i.e., $f : s_i \times \ldots \times s_j \rightarrow s_k$. A special boolean sort is considered, $\mathcal{B}$, with the constants *true* and *false* as elements. Boolean classical operators like $\wedge, \vee, \rightarrow$ are regarded as functions of type $f : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ and $\neg$ as $f : \mathcal{B} \rightarrow \mathcal{B}$. Each predicate $P(t1_{s1}, \ldots, tn_{sn})$ has associated a function mapping terms $t1_{s1}, \ldots, tn_{sn}$ from sorts $s1, \ldots, sn$ to $\mathcal{B}$. In particular, we could interpret the symbol $\doteq$ in this way.

These models are restricted by conditions imposed to particular relations introduced in the previous section. For example, the relation $begin, end : \mathcal{I} \rightarrow \mathcal{T}$ restricts the possible models to those where the beginning of an interval is smaller than its end. Similarly, $Begin_{\mathcal{E}}, End_{\mathcal{E}} : \mathcal{D} \rightarrow \mathcal{N}$ restricts valid models to those stisfying that the starting event of a durative event is previous than its ending event. We postpone more details about semantics to other paper so the emphasis of this paper is kept at the syntactic level.

§4 More on Applications

The previous sections describe a theory conceived to handle the notions of time and change. In that sense it can be seen as a specific theory. On the other side it is a general purpose device which can be specialized in many ways. It still can (and must) be supplemented with more specific knowledge in order to solve real problems. Some potential applications were suggested in a previos section and more applications are considered through the examples given below.

Example 4.1

Temporal Databases [35) 33)] deal with problems associated to storing and recovering time-dependent information. There are several proposals to build this kind of systems which can be grouped into two main approaches, the extension of the relational model and the extension of the deductive model [37)]. The deductive approach resorts to a declarative specification of the information through facts (explicit data) and rules (implicit data). $\mathcal{L}^{\mathbb{T}}$ can be used both as a language to store temporal information and to query the database as it is usual in the deductive approach to databases. Having a declarative temporal language gives also a very powerful and convenient way to state database constraints. [15)] Recently some awareness has arisen about the need of being able to refer both to instantaneous as well as durative temporal references for an appropriate handling of temporal data in this context.

Previous works about point-based approaches in the literature are numerous and show how advantageous is to have a point-based set of primitives. [36)] On the other hand, recent research shows that duration based notions are equally helpful and sometimes essential to capture some situations of the real world [34)]. A situation where duration related primitives are essential and could not be replaced by a point-based mapping involves reasoning about events occurrence. This is because there are non-homogeneous events, i.e. their holding in an interval I does not allow to infer it occurred in a subinterval of I. An example of these events is a phone call performed from A to B during $[I_1, I_2]$. The same phone call cannot be said to have occurred in a subinterval of $[I_1, I_2]$ neither it can be decomposed in any serie of calls taking the same period. That series will have different properties, for example that of being more expensive.

Example 4.2

This problem deals with the possibility and necessity to combine effects of actions in order to achieve a unique goal. The scenario described in [5)] includes the attempt to decouple a car by activating the decoupler while the engine is moving forward. A way to write Allen and Fergusons scenario in $\mathcal{L}^{\mathbb{T}}$ would be as follows:

$\text{MEETS}(I_0, I_1) \wedge \text{MEETS}(I_1, I_2)$
$\text{Holds}_{on}(\textit{coupled}(n1, car1), I_0)$
$\text{Do}_{on}(\textit{setthrottle}(n1), I_1)$
$\text{Do}_{on}(\textit{activating}(n1), I_2)$
$(SE1)\ \forall_{\mathcal{W}} n \forall_{\mathcal{I}} I, I' (\text{Do}_{on}(\textit{setthrottle}(n), I) \rightarrow \text{Occurs}_{on}(\textit{move}(n), I'))\ \wedge \text{MEETS}(I, I')$
$(SE2)\ \forall_{\mathcal{W}} n \forall_{\mathcal{I}} I (\text{Do}_{on}(\textit{activating}(n), I) \rightarrow \text{Occurs}_{on}(\textit{activate}(n), I))$
$(SE3)\ \forall_{\mathcal{W}} n \forall_{\mathcal{I}} I (\text{Occurs}_{on}(\textit{move}(n), I) \rightarrow \text{Holds}_{on}(\textit{moved}(n), I))$
$(SE4)\ \forall_{\mathcal{W}} n, c \forall_{\mathcal{I}} I', I'', I (\text{Holds}_{on}(\textit{moved}(n), I') \wedge \text{Occurs}_{on}(\textit{activate}(n), I'') \wedge \neg(I' \lhd\!\rhd I'') \wedge$
$\quad \text{MEETS}(I, \textit{conj}(I', I'') \wedge \text{Holds}_{on}(\textit{coupled}(n, c), I) \rightarrow \text{Occurs}_{on}(\textit{uncouple}(n, c), \textit{conj}(I', I'')))$

where $\textit{conj}(I', I'')$ is the intersection interval between I' and I'' and

$$I \lhd\!\rhd I' =_{def} \text{BEFORE}(I, I') \ \vee\ \text{MEETS}(I, I') \ \vee\ \text{BEFORE}(I', I) \ \vee\ \text{MEETS}(I', I)$$

$(SE5)\ \forall n, c \forall_{\mathcal{I}} I (\text{Occurs}_{on}(\textit{uncouple}(n, c), I) \rightarrow \neg \text{Holds}_{on}(\textit{coupled}(n, c), I') \wedge \text{MEETS}(I, I')$
$\forall_{\mathcal{A}} a \forall_{\mathcal{I}} I (\text{Do}_{on}(a, I) \leftrightarrow ((a = \text{Do}_{on}(\textit{setthrottle}(n1), I) \wedge I = I_1) \vee (a = \text{Do}_{on}(\textit{activating}(n1), I)) \wedge I = I_2)))$

A line of reasoning is that the car will tend to remain coupled getting $\text{Holds}_{on}(\textit{coupled}(n1, car1), I_2)$ by persistency. [8) 12)] The Frame Problem is handled in Allen's proposal using the *Explanation Closure Axioms* technique. This strategy involves to add axioms stating that *no properties change unless explicitly changed by an event* and *no events occur except as the result of the actions.*

If setting the throttle occurs later, for instance $\text{Do}_{on}(\textit{setthrottle}(n1), I_2)$, it is not possible to use SE1 and SE3 and the coupling persists. But, if the moving event occurs during I_2 the decoupling can be achieved as indicated by SE1:

$$\text{Do}_{on}(\textit{setthrottle}(n), I_1) \rightarrow \text{Occurs}_{on}(\textit{move}(n), I_2))$$

From $\text{Do}_{on}(\textit{activating}(n1), I_2)$ and SE2 we can obtain $\text{Occurs}_{on}(\textit{activate}(n1), I_2)$:

$$\text{Do}_{on}(\textit{activating}(n), I_2) \rightarrow \text{Occurs}_{on}(\textit{activate}(n), I_2))$$

By persistency it could be assumed $\text{Holds}_{on}(\textit{coupled}(n1, car1), I_1)$ then we could support $\text{Occurs}_{on}(\textit{uncouple}(n1, car1), I_2)$ by SE4 as follows:

$$\begin{aligned} \text{Holds}_{on}(\textit{move}(n1), I_2) \wedge \quad & \text{Occurs}_{on}(\textit{activate}(n1), I_2) \wedge \\ & \neg(I_2 \lhd\!\rhd I_2) \wedge \text{MEETS}(I_1, I_2) \wedge \\ & \text{Holds}_{on}(\textit{coupled}(n1, car1), I_1) \rightarrow \quad \text{Occurs}_{on}(\textit{uncouple}(n1, car1), I_2)) \end{aligned}$$

Finally, see figure 3, by $SE5$ we can conclude $\neg\text{Holds}_{on}(\textit{coupled}(n1, car1), I_2)$:

$$\text{Occurs}_{on}(\textit{uncouple}(n1, car1), I_2) \rightarrow \neg\text{Holds}_{on}(\textit{coupled}(n1, car1), I_2)$$

But because Allen's interval relations are qualitative in essence several restrictions to desirable practical reasoning apply involving even simple quantitative reasoning.

a) if we want to specify that the action *setthrott le* must last more than two time units in order to cause the move or that the action *activating* must be started earlier than one unit after the start of *setthrotle* we can use the following rules in $\mathcal{L}^{\mathbb{T}}$:

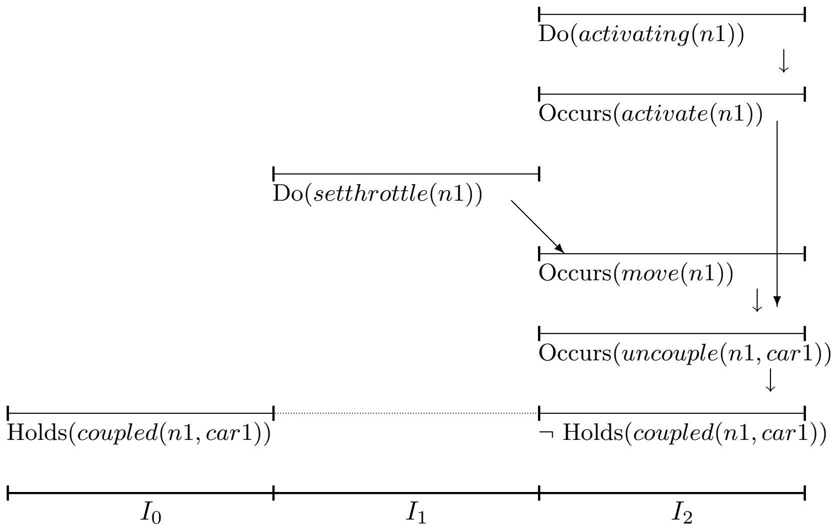


Fig. 3 Synergistic Effects, a way to obtain $\neg\text{Holds}_{on}(coupled(n1, car1), I_2)$

$(SE1')\forall_{\mathcal{W}} n \forall_{\mathcal{I}} I, I' (\text{Do}_{on}(setthrottle(n), I) \wedge length(I) \geq 2 \rightarrow \text{Occurs}_{on}(move(n), I')) \wedge \text{MEETS}(I, I')$

$(SE2')\forall_{\mathcal{W}} n \forall_{\mathcal{I}} I, I' (\text{Do}_{on}(setthrottle(n), I') \wedge \text{Do}_{on}(activating(n), I) \wedge$
$\text{OVERLAPS}(I', I) \wedge length([begin(I'), begin(I)]) < 1 \rightarrow \text{Occurs}_{on}(activate(n), I))$

but there is no way to state that in Allen's proposal.

b) Could not be possible to take any decisions based on subtler issues as determining if the decoupling event is or not instantaneous. For example, this can be seen when the behaviour of a device depends on the time that the decoupling takes. This device would be required to differentiate between an instantaneous decoupling, which is an abrupt and risky one, and a durative decoupling which indicates a more controled and safe step.

The language $\mathcal{L}^{\mathbb{T}}$ has also been used [8) 9) 11)] as the monotonic layer for a temporal argumentative system able to solve typical problems in Artificial Intelligence like those described in Allen's TRAINS project [5)] and Sandewall's test suite [29)].

§5 Other proposals

There are many proposals in the literature related to temporal notions representation. Many works on philosophy and artificial intelligence are related to our proposal. Because of limitations on space we restrict ourselves to consider some of the most relevant proposals in the literature.

5.1 Van Benthem on the Logic of Time

In what follows we will see how much of vanBenthem's proposal [38) chapter I.4] could be reconstructed on the basis of our previous definitions. Let us first consider the following abbreviations:

$$xOy =_{def} \exists z (z \sqsubseteq x \wedge z \sqsubseteq y)$$

$$xUy =_{def} \exists z (x \sqsubseteq z \wedge y \sqsubseteq z)$$

and the function $\mathrm{U} : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ where $\mathrm{U}([x_1, x_2], [y_1, y_2]) =_{def} [min(x_1, y_1), max(x_2, y_2)]$ If we also consider the following axioms as the axiomatization for a convex set of intervals derived from a point structure (theorem I.4.1.4):

$(\lessdot, \mathrm{TRANS}) : \forall I_1, I_2, I_3(I_1 \lessdot I_2 \lessdot I_3 \to I_1 \lessdot I_3)$ $(\lessdot, \mathrm{IRREF}) : \forall I_1 \neg(I_1 \lessdot I_1)$
$(\sqsubseteq, \mathrm{TRANS}) : \forall I_1, I_2, I_3(I_1 \sqsubseteq I_2 \sqsubseteq I_3 \to I_1 \sqsubseteq I_3)$
$(\sqsubseteq, \mathrm{REF}) : \forall I_1(I_1 \sqsubseteq I_1)$
$(\sqsubseteq, \mathrm{ANTIS}) : \forall I_1, I_2(I_1 \sqsubseteq I_2 \sqsubseteq I_1 \to I_1 \cdot\!=\!\cdot I_2)$
$(\sqsubseteq, \mathrm{CONJ}) : \forall I_1, I_2(I_1 O I_2 \to \exists I_3(I_3 \sqsubseteq I_1 \wedge I_3 \sqsubseteq I_2 \wedge \forall I_4(I_4 \sqsubseteq I_1 \wedge I_4 \sqsubseteq I_2 \to I_4 \sqsubseteq I_3)))$
$(\sqsubseteq, \mathrm{DISJ}) : \forall I_1, I_2(I_1 U I_2 \to \exists I_3(I_3 \sqsupseteq I_1 \wedge I_3 \sqsupseteq I_2 \wedge \forall I_4(I_4 \sqsupseteq I_1 \wedge I_4 \sqsupseteq I_2 \to I_4 \sqsupseteq I_3)))$
$(\sqsubseteq, \mathrm{FREE}) : \forall I_1, I_2(\forall I_3(I_3 \sqsubseteq I_1 \to I_3 O I_2) \to I_1 \sqsubseteq I_2$
$(\sqsubseteq, \mathrm{DIR}) : \forall I_1, I_2 \exists I_3(I_1 \sqsubseteq I_3 \wedge I_2 \sqsubseteq I_3)$
$(\sqsubseteq, \mathrm{ATOM}) : \forall I_1 \exists I_2(I_2 \sqsubseteq I_1 \wedge \forall I_3(I_3 \sqsubseteq I_2 \to I_3 \cdot\!=\!\cdot I_2))$
$(\lessdot, \sqsubseteq, \mathrm{MON}) : \forall I_1, I_2((I_1 \lessdot I_2 \to \forall I_3(I_3 \sqsubseteq I_1 \to I_3 \lessdot I_2)) \wedge (I_1 \lessdot I_2 \to \forall I_3(I_3 \sqsubseteq I_2 \to I_1 \lessdot I_3)))$
$(\lessdot, \sqsubseteq, \mathrm{MOND}) : \forall I_1, I_2((I_1 \lessdot I_2 \to \forall I_3(I_3 \lessdot I_2 \to \mathrm{U}(I_1, I_3) \lessdot I_2)) \wedge (I_1 \lessdot I_2 \to \forall I_3(I_2 \lessdot I_3 \to I_2 \lessdot \mathrm{U}(I_1, I_3))))$
$(\lessdot, \sqsubseteq, \mathrm{CONV}) : \forall I_1, I_2, I_3(I_1 \lessdot I_2 \lessdot I_3 \to \forall I_4((I_1 \sqsubseteq I_4 \wedge I_3 \sqsubseteq I_4) \to I_2 \sqsubseteq I_4))$

Lemma 5.1
$(\lessdot, \mathrm{TRANS})$, $(\lessdot, \mathrm{IRREF})$, $(\sqsubseteq, \mathrm{TRANS})$, $(\sqsubseteq, \mathrm{REF})$, $(\sqsubseteq, \mathrm{ANTIS})$, $(\sqsubseteq, \mathrm{CONJ})$, $(\sqsubseteq, \mathrm{DISJ})$, $(\sqsubseteq, \mathrm{FREE})$, $(\sqsubseteq, \mathrm{DIR})$, $(\sqsubseteq, \mathrm{ATOM})$, $(\lessdot, \sqsubseteq, \mathrm{MON})$, $(\lessdot, \sqsubseteq, \mathrm{MOND})$ and $(\lessdot, \sqsubseteq, \mathrm{CONV})$ are theorems in $\mathcal{E}_x$.

Proof $(\lessdot, \mathrm{TRANS})$: if $x \lessdot y \lessdot z$, by definition of interval, points defining x are before than those of z.
$(\lessdot, \mathrm{IRREF})$: points defining an interval cannot be before than themselves by irreflexivity on INS.
$(\sqsubseteq, \mathrm{TRANS})$: from the relation of beginning and ending points stated on the hypothesis, transitivity over $\mathcal{T}$ and the definition of $\sqsubseteq$.
$(\sqsubseteq, \mathrm{REF})$: by considering $begin(I_1)$ and $end(I_1)$, the order relation over time points and definiton of $\sqsubseteq$.
$(\sqsubseteq, \mathrm{ANTIS})$: reasoning by cases over the hypothesis, we arrive to the only consistent choice of $\cdot\!=\!\cdot$ on beginning and ending points for I_1 and I_2.
$(\sqsubseteq, \mathrm{CONJ})$: by definition of O we know there exists a common interval, let us call it z, then $z \sqsubseteq x \wedge z \sqsubseteq y$ is true and so $\forall I_4(I_4 \sqsubseteq I_1 \wedge I_4 \sqsubseteq I_2 \to I_4 \sqsubseteq z)$ indicating that z is the biggest for these x and y. Then it suffices to take the biggest one satisfying xOy.
$(\sqsubseteq, \mathrm{DISJ})$: the dual of CONJ
$(\sqsubseteq, \mathrm{FREE})$: if all subinterval of x overlaps with y it is impossible to consider a subinterval of x without overlapping y, therefore $x \sqsubseteq y$.
$(\sqsubseteq, \mathrm{DIR})$: is derivable from the definition of interval (the fact that two points always define an interval).
$(\sqsubseteq, \mathrm{ATOM})$: follows the definition of $\sqsubseteq$ and the fact that the smaller subinterval allowed by definition is of equal length to one, reserving zero-length elements to instants.
$(\lessdot, \sqsubseteq, \mathrm{MON})$: we have considered only the first half. If we get by hypothesis that $I_1 \lessdot I_2$ and if $I_3 \sqsubseteq I_1$ then by definition of $<$ and $\sqsubseteq$ we conclude $I_3 \lessdot I_2$. The dual is analogous.
$(\lessdot, \sqsubseteq, \mathrm{MOND})$: we consider the first half. If we get by hypothesis that $I_1 \lessdot I_2$ and if $I_3 \lessdot I_2$ then by definition of $<$ and function U we conclude that the interval $\mathrm{U}(I_1, I_3)$ is before I_2. The dual is analogous.
$(\lessdot, \sqsubseteq, \mathrm{CONV})$: by definition of interval, $<$ and $\sqsubseteq$. ■

OBSERVATION: $(\lessdot, \text{ASYM}) : \forall I_1, I_2(I_1 \lessdot I_2 \rightarrow \neg(I_2 \lessdot I_1))$ could be derived directly because if $I_1 \lessdot I_2$ then by definition of interval $end(I_1) < begin(I_2)$, and by definition of relation $\lessdot$ over intervals we had $\neg(I_2 \lessdot I_1))$. Alternatively, it could be derived from $TRANS$ and $IRREF$.

As the reader can see, we have refused Van Benthem's suggestion to include $(\lessdot, \sqsubseteq, \text{FREE}^\star)$, which in our notation becomes:

$$\forall I_1, I_2(\neg(I_1 \lessdot I_2) \rightarrow \exists I_3, I_4((I_3 \sqsubseteq I_1 \wedge I_4 \sqsubseteq I_2) \rightarrow \forall I_5 \forall I_6(I_5 \sqsubseteq I_3 \wedge I_6 \sqsubseteq I_4 \rightarrow \neg I_5 \lessdot I_6)))$$

because it does not hold in our framework if we consider two abutting intervals. A theorem we could get from previous axiomatization is the extension of discreteness to intervals:

Lemma 5.2
$\forall I_1 \exists I_2(I_2 \sqsubseteq I_1 \wedge \forall I_3(I_3 \sqsubseteq I_2) \rightarrow I_3 \cdot\!\!=\!\!\cdot I_2))$

Proof To see that it follows from our previous 6 and 7 let us consider three points i_1, i_2, i_3. It is enough to take $I_1 \cdot\!\!=\!\!\cdot [i_1, i_3]$ and $I_2 \cdot\!\!=\!\!\cdot [i_1, i_2]$ or $I_2 \cdot\!\!=\!\!\cdot [i_2, i_3]$. ■

As it is clear from $(\lessdot, \sqsubseteq, \text{CONV})$ we are considering convex intervals here. That is, we are considering a set of intervals:

$$CONV(\mathcal{I}) =_{def} \begin{array}{l}\{c = [i_1, i_2] | \text{ for all interval } s = [s_1, s_2] \text{ such that} \\ \text{if } c_1 < s_1 < s_2 < c_2 \text{ then } s \sqsubseteq c\}\end{array}$$

In addition to the previous definitions we could consider the relation $OVER(\mathcal{I}) \subseteq CONV(\mathcal{I}) \times CONV(\mathcal{I})$ of overlapping convex intervals:

$$OVER(\mathcal{I}) =_{def} \{(c_1, c_2) | c_1 = [i_1, i_2], c_2 = [i_3, i_4], (i_1 \leq i_3 < i_2) \text{ or } (i_3 \leq i_1 < i_4)\}$$

We had considered restricted convex intervals to assure that the functions to be given now are totally defined. Let $CONJ : OVER(\mathcal{I}) \rightarrow \mathcal{I}$ as $CONJ([i_1, i_2], [i_3, i_4]) =_{def} [max(i_1, i_3), min(i_2, i_4)]$ (min and max have the intended meaning over numbers) and $DISJ : OVER(\mathcal{I}) \rightarrow \mathcal{I}$ as $DISJ([i_1, i_2], [i_3, i_4]) =_{def} [min(i_1, i_3), max(i_2, i_4)]$. Based in this definitions we have the following results (compare with vanBenthem's [38) pp. 63]):

Lemma 5.3
$\langle OVER(\mathcal{I}), CONJ, DISJ \rangle$ is a (non-distributive) lattice.

Proof Let us consider the following intervals $X = [x_1, x_2], Y = [y_1, y_2], Z = [z_1, z_2]$. It is easy to see that $CONJ$ and $DISJ$ are closed on $OVER(\mathcal{I})$ from their definitions. $\langle OVER(\mathcal{I}), CONJ, DISJ \rangle$ has the following properties:

Commutativity: $DISJ(X, Y) \cdot\!\!=\!\!\cdot DISJ(Y, X)$ because is always chosen the minimum left point as the left bound and the maximum of the right points as the right bound and both operations are commutative.

Associativity: $DISJ(DISJ(X, Y), Z) \cdot\!\!=\!\!\cdot DISJ(X, DISJ(Y, Z))$ because, again, by commutativity of min and max we get the leftmost point from the beginning of X,Y and Z regardless of the order of evaluation. Similarly we get the rightmost point.

Absorption: $DISJ(X, CONJ(X,Y)) \cdot\!=\!\cdot X$ because by CONJ we obtain an interval that is a subinterval of X and by definition of DISJ we obtain X. It is straightforward to prove that the duals of these properties also hold.

To see that it is not distributive it is enough to see that $CONJ(Y, Z)$ could not overlap with another convex interval X and in that case DISJ would not be defined. In the special case in which $DISJ(X, Y)$ overlaps with $DISJ(X, Z)$ then there must be a common subinterval. Then, the first term will have all X and the common part of Y and Z while in the second term we will have X in all DISJ operations and the common part of Y and Z by CONJ leading to the same interval. ■

Observation: we could not prove that there is a Boolean Algebra as vanBenthem [38) page 81] did, because in doing so we must prove that $CONJ(\emptyset, X) = \emptyset$. This does not apply here because there is not such a thing as an empty interval in our framework.

5.2 Kamp and Thomason on the Logic of Events

There had been many attempts to formalize the notion of events in logical frameworks. The purpose of this section is to explain why some of those attempts where not adecuate for this proposal. Here we shall consider two well-known of those attempts starting with Kamp's proposal [23)].

Definition 5.1
The *event structure* $\mathcal{K} : (K, <_k, O_k)$ consist of a set K of events and two binary relations: $<_k, O_k$. $<_k \subseteq (K \times K)$ is an order relation between elements of K. $O_k \subseteq (K \times K)$ is a relation used to denote overlapping between elements of K. The relations $<_k, O_k$ satisfy the following axioms:

$$
\begin{array}{l}
k_1 <_k k_2 \rightarrow \neg(k_2 <_k k_1) \\
k_1 O_k k_2 \rightarrow k_2 O_k k_1 \\
k_1 O_k k_1 \\
k_1 <_k k_2 \rightarrow \neg k_1 O_k k_2 \\
k_1 <_k k_2 \wedge k_2 O_k k_3 \wedge k_3 <_k k_4 \rightarrow k_1 <_k k_4 \\
k_1 <_k k_2 \oplus k_1 O_k k_2 \oplus k_2 <_k k_1
\end{array}
$$

where $A \oplus B =_{def} (A \vee B) \wedge \neg(A \wedge B)$

Observation: Using axiom $k_1 <_k k_2 \oplus k_1 O_k k_2 \oplus k_2 <_k k_1$ it could be defined, in the same way as was done for sort $\mathcal{E}_v$, a simultaneity notion between events in the following way:

$$k_1 S_k k_2 =_{def} (\neg k_1 <_k k_2) \wedge (\neg k_2 <_k k_1)$$

Using the notion of event recently defined, Kamp considers the temporal notion associated without resorting to explicit time references.

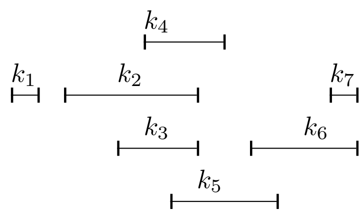
Definition 5.2
Let $\mathcal{K} : (K, <_k, O_k)$ an event structure. An *instant*, i, is a maximal subset of K such that:

1. $\forall k_1, k_2 \in i\ (k_1 O_k k_2)$
2. $\forall (k_1 \in K - i)\ \exists\ k_2 \in i\ \neg(k_1 O_k k_2)$

Let $I(K)$ be the set of instants associated to K and $i, i' \in I(K)$. The notion of a precedence order over instants could be reconstructed from that defined for events (see lemma 2.4):

$$i <_{ik} i' =_{def} \exists k \in i, k' \in i'(k <_k k')$$

Example 5.1
Let the set of events $K : \{k_1, k_2, k_3, k_4, k_5, k_6, k_7\}$ satisfying the following relations:



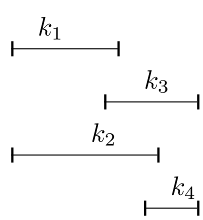
Instants $i_1 = \{k_1\}$, $i_2 = \{k_4, k_2, k_3, k_5\}$, $i_3 = \{k_6, k_5\}$ and $i_4 = \{k_7, k_6\}$ could be defined from these events and it is true that $i_1 <_{ik} i_2 <_{ik} i_3 <_{ik} i_4$.

It is important to bear in mind that if we consider a stronger order, for example:

$$i_1 \lhd i_2 =_{def} \forall k_1 \in i_1, k_2 \in i_2(k_1 <_k k_2)$$

we could have instants that are not before another but are not truly equal because they do not satisfy the definition of instant. For example, $i_2 \not\lhd i_3$ and $i_3 \not\lhd i_2$, then we must accept $i_2 S_k i_3$. But, it is difficult to think of i_2 and i_3 as denoting the same temporal space because they do not constitute the same instant as a set (event k_6 does not overlap with those of i_2).

Observation: As we could observe an event could be part of more than one instant, depending on the way they are grouped. If we consider the previous example, event k_5 is part of i_2 and i_3 also k_6 is part of i_3 and i_4. In other words, they could overlap. This seems rather conflictive with the notion of an instant because they are supposed atomic, in some sense, and therefore they are usually assumed as either distinct or equal. Kamp's proposal seems to be more suitable to the concept of a period. We can perceive also the problem in the following way. Let the instants $i_1 = \{k_1, k_2, k_3\}$ and $i_2 = \{k_2, k_3, k_4\}$ be in the following arrangement:



As we could see, by definition we have that $i_1 <_{ik} i_2$ because $k_1 <_k k_4$. However, both instants have the same temporal extension as a whole, which is counterintuitive with the intended order relation. Thomason's proposal considers a structure of events $\mathrm{W} = (W, \prec, \prec_0, \prec_1)$, where "$\prec_0$" must be read as "begins before" and "$\prec_1$" as "ends before", satisfying axioms:

$$w \not\prec w$$
$$w_1 \prec w_2 \wedge w_3 \prec w_4 \rightarrow w_1 \prec w_4 \vee w_3 \prec w_2$$
$$w_1 \prec_0 w_2 \leftrightarrow \exists w(w \prec w_2 \wedge w \not\prec w_1)$$
$$w_1 \prec_1 w_2 \leftrightarrow \exists w(w_1 \prec w \wedge w_2 \not\prec w)$$

In addition $w_1 O_t w_2 =_{def} w_1 \not\prec w_2 \wedge w_2 \not\prec w_1$.

Later, Thomason [32] offered another event-based framework considering the relation "Abuts". Also this definition is unsatisfactory to us but, for different reasons from the previous one. Although technically better than the previous proposal there are reasons to avoid adopting it in this context. His work is addressed to enhance Rusell's proposal [28]. Such goal influences his formalization in an undesirable way as the author himself comments. [31) pp. 86] Furthermore, he starts assuming durative events, which is clear from the basic relations. From this it seems that his proposal, as Kamp's, are more suitable as a formalization of periods of time and not of "periods from instants" as in our case.

5.3 Bochman's Theory of Instants and Intervals

One of the best-known proposals in the temporal logic literature that considers an ontology based both on instants and intervals is that offered by Bochman. [13] [14] In this section we will show how its definitions and axioms are embeded in our theory.

With respect to his definitions it is enough to see they can be rewritten in our notation as follows: $i \prec I$ indicates that "the point i is part of interval I" is sinthesized in our predicate $In(i, I)$. His notion of "an interval being part of another", $I_1 \preceq I_2$, is represented in $\mathcal{L}^{\mathbb{T}}$ as $I_1 \sqsubseteq I_2$ while the overlapping between intervals, $I_1 \circ I_2$, becomes $OVERLAPS(I_1, I_2)$ in our framework. Finally, an interval limited by points, $C(i_1 I\, i_2)$, is expressed through $I \cdot\!\!=\!\!\cdot [i_1, i_2]$. After giving these definitions, Bochman follows with a series of axioms relating points with intervals (we will reproduce them using our notation):

PI$_1$ $\forall_{\mathcal{I}} I_1, I_2 \forall_{\mathcal{T}} i(In(i, I_1) \leftrightarrow In(i, I_2)) \rightarrow I_1 \cdot\!\!=\!\!\cdot I_2$

PI$_2$ $\forall_{\mathcal{T}} i_1, i_2(i_1 \neq i_2 \rightarrow \exists I(begin(I) \doteq i_1 \wedge end(I) \doteq i_2))$

PI$_3$ $\forall I \exists i_1, i_2(begin(I) \doteq i_1 \wedge end(I) \doteq i_2)$

PI$_4$ $\forall I_1, I_2, i_1((In(i_1, I_1) \wedge In(i_1, I_2)) \rightarrow \exists I_3 \forall i_2(In(i_2, I_3) \leftrightarrow (In(i_2, I_1) \vee In(i_2, I_2)))$

PI$_5$ $\forall i_1, i_2, i_3(i_1 \neq i_2 \neq i_3 \rightarrow \exists I(In(i_1, I) \wedge \neg(In(i_2, I) \leftrightarrow In(i_3, I)))$

PI$_6$ $\forall i \exists I_1, I_2(In(i, I_1) \wedge In(i, I_2) \wedge \neg OVERLAPS(I_1, I_2)$

Lemma 5.4

Axioms $\mathrm{PI}_1, \mathrm{PI}_2, \mathrm{PI}_3, \mathrm{PI}_4, \mathrm{PI}_5, \mathrm{PI}_6$ are provable in $\mathcal{L}^{\mathbb{T}}$.

Proof We proceed by showing how to obtain the same axioms in our proposal.

PI_1 : if $\forall_{\mathcal{I}} I_1, I_2 \forall_{\mathcal{T}} i(In(i, I_1) \leftrightarrow In(i, I_2))$, in particular i can range on a set of numbers whose minimum are $begin(I_1)$ and $begin(I_2)$ and we can deduce $begin(I_1) \doteq begin(I_2)$. Analogously, as i can obtain as a maximum $end(I_1)$ and $end(I_2)$ we deduce $end(I_1) \doteq end(I_2)$. Putting these two facts together we obtain $I_1 \cdot\!\!=\!\!\cdot I_2$ because of the definition in 2.1.

PI_2 : by definition 2.3 an interval is a pair of points i_1, i_2 such that $i_1 < i_2$.

PI_3 : by definition 2.4 we have: $\forall I\, \exists i$ begin$(I) \doteq i$ and $\forall I\, \exists i$ end$(I) \doteq i$

PI_4 : the axiom expresses that all pair of intervals with at least a point in common defines an interval that contains them. According to the hypothesis and definitions given in 2.1 we can deduce that $OVERLAPS(I_1, I_2) \vee MEETS(I_1, I_2)$. Then we can build an interval I_u such that

$$I_u = min(begin(I_1), begin(I_2)), max(end(I_1), end(I_2))$$

then I_u clearly takes I_1 and I_2:

1. $\forall i_2(In(i_2, I_u) \rightarrow (In(i_2, I_1) \vee In(i_2, I_2))$
2. $\forall i_2((In(i_2, I_1) \vee In(i_2, I_2) \leftarrow In(i_2, I_u))$

PI_5 : for all triple of points i_1, i_2, i_3 it is always possible to build an interval containing two of these points but not the remainding. If we assume $i_1 \neq i_2 \neq i_3$ for axiom 5 we have a strict order, let us suppose $i_2 < i_1 < i_3$, then we can define $I = [i_2, i_1]$ or $I = [i_1, i_3]$.
If $I = [i_2, i_1]$: $i_3 > end(I)$, consequently we have $\neg In(i_1, I) \wedge In(i_2, I)$
If $I = [i_1, i_3]$: $i_2 > begin(I)$, we have $\neg In(i_2, I) \wedge In(i_3, I)$

PI_6 : consider i as the meeting point between two touching intervals: $end(I_1) = i = begin(I_2)$ ■

As all the remaining concepts over intervals (Bochman's definitions 1 to 4 on page 406) and axioms for interval structures $\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_4, \mathrm{I}_5$ are based on the preceeding definitions and axioms so it could be seen that they also follow from our proposal. There are some differences between our proposals as we started from a point based approach and later we built intervals over instants. Also we offered a wider account of agency and temporal related notions.

5.4 Allen on the Logic of Intervals

Much of the work in the temporal reasoning area was motivated by Allen's influential work. Following we compare our proposal with that of Allen [2]. We shall not consider his later works [6] [3] [4] as they were dedicated to clarify some previous ideas without enlarging the theory or the usefulness of his theory in natural language processing and planning without changing those aspects to be considered here. Later [5] Allen published some issues of concern to our framework but we prefer to delay the consideration of the frame problem for another article in spite of its complexity. In examining the proposal we just will consider general axioms discarding those that represent some knowledge in a particular context.

Considering now the axiomatization we show how to get his axioms one by one.

$$H1 : \ \mathrm{HOLDS}(p, T) \leftrightarrow \forall t(IN(t, T) \rightarrow \ \mathrm{HOLDS}(p, t))$$

is provable from our definition of Holds_{on}. We must prove that:

$$\mathrm{Holds}_{on}(p, I) \leftrightarrow \forall_{\mathcal{I}}\ I\ (I \sqsubseteq I' \rightarrow \mathrm{Holds}_{on}(p, i))$$

From $\mathrm{Holds}_{on}(p, I)$ we have by definition that $\forall_{\mathcal{T}}\ i\ (\mathrm{Divides}(i, I) \rightarrow \mathrm{Holds}_{at}(p, i))$ and by definition of interval $\forall_{\mathcal{I}}\ I\ (I \sqsubseteq I' \rightarrow \mathrm{Holds}_{on}(p, i))$. The other half runs by similar argument.

$$H2 : \ \mathrm{HOLDS}(p, T) \leftrightarrow \forall t(IN(t, T) \rightarrow (\exists s IN(s, t) \wedge \ \mathrm{HOLDS}(p, t))$$

One half of H2 could be derived from H1 and the other half is intended to prevent indefinitely intermingled periods [18, pp. 176], which is not applicable in our discrete framework. Allen's H3 is an abbreviation while

$$H4 : \ \mathrm{HOLDS}(not(p), T) \leftrightarrow \forall t(IN(t, T) \rightarrow \neg \ \mathrm{HOLDS}(p, t))$$

is obtained by negation of H1 although we have not adopted $not(p)$ and we just use $\neg$Holds(p,t). Naturally Allen does not have negation over instants. H5 is a theorem of his previous definitions while

$$H6: \text{ HOLDS}(not(not(p)), T) \leftrightarrow \text{ HOLDS}(p,t))$$

discards nested negation as we asked in our section about syntax of the language $\mathcal{L}^{\mathbb{T}}$. H7 is an abbreviation. For events, Allen considers

$$O1: \text{ OCCURS}(e,t) \wedge IN(t',t) \rightarrow \neg \text{ OCCURS}(e,t')$$

Our axiom $\forall_{\mathcal{D}} E\ \forall_{\mathcal{I}} I\ \forall_{\mathcal{T}} i\ (\text{Occurs}_{on}(E,I) \wedge \text{In}(i,I) \rightarrow \neg\text{Occurs}_{at}(E,i))$ is stronger than O1 because if the event cannot occur in an instant it could not occur in an interval because it cannot occur in all the instants that define the interval. While his

$$O2: \text{ OCCURRING}(p,t) \ \rightarrow \exists t'(IN(t',t) \wedge \text{ OCCURRING}(p,t'))$$

has no direct equivalent in our proposal because we do not consider processes as a basic entity. This is one of the main departures of our proposal with respect to Allen's. We think they must be replaced with a construction of events and properties according to the case. [19] Finally, axiom O3 is not considered because it is not general but an exemplification.

About event and agent causality theories [2] two axioms were given for the former concept:

$$O4: \text{ OCCUR}(e,t) \ \wedge \ \text{ECAUSE}(e,t,e',t') \rightarrow \ \text{OCCUR}(e',t')$$
$$O5: \text{ ECAUSE}(e,t,e',t') \rightarrow \ \text{BEFORE}(t,t') \ \vee \ \text{MEETS}(t,t') \ \vee \ \text{OVERLAPS}(t,t') \ \vee \ \text{IN}(t,t') \ \vee \ \text{EQUALS}(t,t')$$

and the following was offered to formalize agent causality:

$$A1: \text{ OCCURING}(\text{ACAUSE}(\text{agent,ocurrence}),t) \rightarrow \ \text{OCCURING}(\text{ocurrence},t)$$

we do not consider this axiom because we do not have processes

$$A2: \ \forall\text{action } \exists!\text{agent, ocurrence (action } = \text{ ACAUSE(agent,ocurrence))}$$

compels actions to be done just by one agent and to generate just one ocurrence. This is unnecessarily restrictive and we prefered not to impose it as it is clear from axiom 20.

$$A3: \text{ OCCUR}(\text{ACAUSE}(\text{agent,event}),t) \rightarrow \ \text{OCCUR}(\text{event},t)$$

One important thing this axiom expresses is that the event will occur as the outcome of the action intended to cause it. The second aspect of it is that the interval of ocurrence must be the same. This is a much more debatable issue and we have adopted a more different view in the axioms above.

In relation to his theory of causality he gave axioms to formalise his notion of intention. We will not consider the topic in this work, because of the strong relation with non-monotonicity we prefer to delay its consideration to another article when a better framework is considered to deal with it.

Also there are some divergences about the properties of event causality. In Allen's own words "... the ECAUSE relation is transitive, anti-symetric, and anti-reflexive", [2) pp. 138] a conception which is in conflict with our axioms 25 and 26. From section 2.6 it seems that in several respects Allen's claim is not true. As simple as the scenario of balls colliding is, he would not be able to prove that the first impact is the cause of itself repeating since it assumes antirreflexivity. Allen would not also conclude that the shocking of one ball with another is a cause for this to shock the other as he assume antysimmetry.

5.5 Vila's proposal

We have selected a functional approach for the definitions given in section 2.1 because of its simplicity but this is not the only way we can set this framework. We could consider relations and a set of axioms to put explicitly what is intended from the above definitions as Vila proposed. [40] In this section we show how Vila's axioms are included in our framework. For example, from def. 2.3 we get:

$$\forall_{\mathcal{I}} I(begin(I) < end(I))$$

$$\forall_{\mathcal{T}} i_1, i_2((i_1 < i_2) \rightarrow \exists_{\mathcal{I}} I(begin(I) \doteq i_1 \wedge \ end(I) \doteq i_2))$$

From definition 2.4

$$\forall I_{\mathcal{I}} \ \exists_{\mathcal{T}} i \ begin(I) \doteq i$$

$$\forall I_{\mathcal{I}} \ \exists_{\mathcal{T}} i \ end(I) \doteq i$$

From the use of a function we get unique points as beginning and ending points for a given interval:

$$\forall_{\mathcal{I}} I \ \forall_{\mathcal{T}} i_1, i_2(begin(I) \doteq i_1 \wedge \ begin(I) \doteq i_2 \rightarrow (i_1 \doteq i_2))$$

$$\forall_{\mathcal{I}} I \ \forall_{\mathcal{T}} i_1, i_2(end(I) \doteq i_1 \wedge \ end(I) \doteq i_2 \qquad \rightarrow (i_1 \doteq i_2))$$

A major difference between our approaches is that Vila considered a dense structure of time while we were comptempt ourseleves starting from a discrete basement. Also in Vilas proposal, instants and intervals are treated from the very beginning at a same level. In $\mathcal{L}^{\mathbb{T}}$ intervals are defined from instants altough both are considered from the very beggining and recognized as equally important. Vila's approach has advantages which he enumerates in his works, ours allows a step by step strategy in the implementation of the system. Also our proposal seems to be given in more detail with respect to the specification of sorts $\mathcal{P}$, $\mathcal{A}$ and $\mathcal{E}_v$ as well as the treatment to dispense to the notions of causality and individuation.

There are also some words to say about the way we address the problem to represent change. That is to say how to represent that a given property P holds over an interval I_1 and do not over another interval I_2 which immediately follows I_1 avoiding to say that there is an instant where nothing could be said or we are forced to recognize that both holds simultaneously. Vila's advice is to left undefined what happens in one of the interval limits, e.g. I_1, and attach the negation of the property from the beginning of the next interval, e.g. I_2. In our case, because we are assuming a discrete framework, it is possible to have at least two ways of dealing with such a situation. It could be assumed that change occurs between two instants, i.e. in their discrete gap. Instead we could be interested in representing change explicitly. In such a case we could dedicate an instant to that purpose. For example, assuming a granularity of minutes and using I_1 and I_2 to represent two consecutive days, it could be considered that $end(I_1) = 01/19/2000, 11:59pm < i < begin(I_2) = 01/20/2000, 00:01am$ where i is the moment of change. This could be explicitly represented as $\text{Occurs}_{at}(day\_change, i)$.

5.6 A Comparison with Previous Approaches

We can summarize this section higlighting the differences between all these proposals and that which is proposed in this work. All the previous approaches are intended to provide a theory of time which considers concepts like instants, intervals and events as a ground for a theory of change. Some of them come from a purely logic perspective while others where offered in the contex of Artificial Intelligence. The goal of this work is to provide a cohesive framework taking these previous works as a basis. The second important goal is to define a framework which is specified in more detail benefiting later theoretical studies and also being more informative to those who are interested on its implementation. Differences between this proposal and those reviewed in previous section are below highlighted to complement those comments at the end of each different proposal.

Van Benthem's work on the logic of time [38] gives a formal account on temporal ontologies and languages. One choice he explores is that of considering points and intervals as a basis of a theory of time. Unfortunately his analysis, however interesting, does not offer a final proposal for a logic on these lines but it is, in his own words, "suggestive". It would be interesting to analize how to define a logic with all the standard components taking Van Benthem's suggestions as the starting point.

Kamp's and Thomason's proposals on events where considered in a previous section and then it was explained why they are different from our proposal. We also provide a whole framework where the theory of events fits. There are many ocassions where events order and duration are the only available information usually connected to some extent with an incomplete source of information. For example, if it is noticed that some machine stopped because it ran out of raw material. We need to be able to order those facts in time even if we do not know when they exactly occurred. Also we must be able to consider that some event like the machine stop occurred during another event, *e.g.*, an alarm sounding, without being forced to know neither how much they lasted nor even when they occurred. In $\mathcal{L}^{\mathbb{T}}$ events can play an important role in case temporal information is provided without an explicit time attached. This still gives the system a chance to do temporal reasoning while it could make some of the above mentioned proposals useless.

Bochman explored an instant and interval based ontology [13] [14] as well as different ways to combine those different temporal references, leading the author to suggest these features could be appropiate to define an intuitionistic logic. This work has been done from a purely logic perspective and it would be worthy to devise how to supplement it with notions like those considered in this work. A deeper consideration of this work in the context of computer science would require a clearer and more detailed specification of the system regarding syntax, semantics, inference rules, events, actions and persistency.

The interval logic for temporal reasoning proposed by Allen [2] [5] has been one of the most influential in the literature of the associated field in AI. However its specification lacks some level of detail that would allow to consider its implementation. To cite some of these features we can start with some ontological ones of the proposal. There are some details about the temporal structure itself that are intentionally ignored. [2] The author contents himself with a linear conception of time, in pp. 131, and he does not say for example, if time is discrete, dense or continuous, because his aim is to define a general theory of action and time. The lack of specification of these basic features leaves several problematic questions open to fully understand the theory, not to mention those who want to do an implementation of that system. Since it depends on the temporal structure which is choosen if it can be axiomatized in a first-order or in a second-order logic. Because Allen's proposal [2]page 128 is a typed first order logic so it could be inferred that it will be impossible to use a temporal domain which is isomorphic to the real numbers, the naturals or the integers as all of them demand second order axioms in their formalizations. Also nothing is said about different problems which arise naturally once you choose a given temporal structure. For example, once density is allowed something must be said about the "dividing instant problem". [38) page 4] Instead we considered in this work a proposal which is more on the line of a detailed approach, specifying in detail what the temporal hypothesis of $\mathcal{L}^{\mathbb{T}}$ are. We adopted a discrete and entirely first-order axiomatizable temporal structure which acts as a departing point of our proposal. This allows future extensions of the system which would consider dense or continuous time in an ordered and progressive way. Regarding the logic system as a whole we can also notice that some basic and desirable ingredients are lacking, such as 1) a clear specification of the syntax and semantic 2) the inference rules 3) a detailed axiomatization of any concept different from intervals.

Vila's work is one of the latest approaches in the field including instants and intervals as a part of the temporal ontology. In fact much of that work was a source of inspiration for this proposal. There are some differences however. While his proposal is based on a dense conception of time we started from a purely discrete structure. The Presentation is also different at the level of depth we consider events, actions, causality, inference rules and individuality. One major concern of Vila's work was the *Dividing Instant Problem* as a result of adopting a dense temporal structure. It was shown in the last section that it is not a problem for our proposal as we are considering a discrete approach.

§6 Conclusions

Many applications need to handle the notion of time and change at some extent. The scenarios where the system is intended to work can demand the representation of temporal information in different ways. Typically we could need to consider instantaneous or durative references as well as quantitative or qualitative temporal information. Several proposals have been made to incorporate theories which can handle these features. However, most of them need more clarification regarding some basic features in order to be considered a good departure point for an implementation.

Throughout this article a general framewok for representing and reasoning with temporal information is provided which enhances previous proposals in several ways:

1. we start from a well-known many-sorted logic with functions and equality which has a Gentzen-style proof system with a resolution method associated.
2. the considered sorts are clearly specified. In the particular case of events, we provided axioms and definitions that allow us to reason without explicit time. This reduces the impact of lacking information. Also a connection between explicit time and event-based reasoning is established.
3. the temporal ontology is presented in two steps that allow incremental implementation, starting with non-durative temporal notions and building durative temporal references from them.
4. the consideration of both instants and intervals provides a way to solve problems, which where addressed elsewhere. [18)]
5. because in our proposal most of the temporal constraint problems could be translated as purely instant-based or instant-interval based, it is more likely to have more efficient implementation. We could take advantage of previous research [27)] where algorithms were proposed for that kind of constraint problems. These algorithms will give us more efficiency in the temporal constraint solver than in case we are forced to do constraint reasoning in a purely interval-based framework [41)].

It is worth mentioning that temporal reasoning seems not to be an isolated phenomena in rational process and many other kinds of reasoning need to be considered in relation with it. Here just a few of them have been selected leaving to future work the exploration of concepts as granularity, spatio-temporal reasoning, deeper considerations of qualitative reasoning, the frame problem and non-monotonic reasoning. Some of these problems were considered by the author in complementary work. The implementation of the system has not been done yet but it seems to be naturally carried out in a typed programming language where the different sorts could be directly represented.

This logic, supplemented with the explanation closure technique, [30] was used in connection to a nonmonotonic meta-system [8] [9] [11] [12] solving succesfully well-known problems of the artificial intelligence literature [5] [29]. It is also posible to use it alone to solve problems in the wide spectrum of areas that need ways to formalize the notion of change, e.g. databases, multimedia, real-time, scheduling, natural language, to name just a few. The adequacy of the proposed framework to particular areas and classes of problems is a present line of research. The above list of features makes it attractive by itself and the examples included in the last section show its versatility.

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