

CHAPTER 1

CHOOSING THE FRAMEWORK

The material presented in this chapter can be viewed as auxiliary to the main body of work. While the main target of the present research is to develop a logic that adequately represents evidential setup the author's main concern was to explore the formal properties of the proposed approach. As the research on the topic progressed and the first results were presented the question 'why this particular logic?' were posed by several member of the research community who were otherwise happy with the presented findings. A look at the materials recently and not so recently published on the topic also reveals that there is no consensus on what is the preferred logic framework for addressing the topic.

The author would not go as far back as finding a justification for using the fuzzy logic apparatus for formalizing decision making and reasoning under uncertainty. The scope and purpose of this chapter is more modest. Assuming that the readership agrees with the idea of using fuzzy logic for inference within the Dempster-Shafer theory, I attempt to show that a superintuitionistic logic as opposed to a modal or Boolean suits the purpose the best.

Even though there are several purely formal reasons for choosing the approach that was taken, there are also factors of more general nature that influenced the author's decision. Below, are the philosophical and mathematical considerations that influenced my choice.

The chapter is organized as follows. We start with a short exposition of the major philosophical approaches to understanding the nature of mathematical objects. Based on this exposition I explain why I prefer Brouwer's approach as applied to the problem at hand. Next, the formal consequences of the choice are discussed and the final choice is made.

1.1 The nature of mathematical objects

1.1.1 Historical Remarks

This work is devoted to the development of a mathematical formalism that is used for reasoning within the Dempster-Shafer theory. Thus the subject of this inquiry is a collection of mathematical objects. Given the applied nature of the belief theory, the ability to update the knowledge, possibly as a result of the interaction with the outside world, is important. The collection of the mathematical objects that is being constructed should not be independent from the notion of time. Discussing the issues of temporality of the mathematical objects cannot be done without a short foray into the philosophy of mathematics. Below, we take a look at the relevant definitions of a mathematical object and its basic properties.

The discussion about the nature of mathematical objects is ages old, but we confine our discussion to the formalisms that date back to XIX and XX centuries. The author believes that the works that have shaped the modern understanding of what a mathematical object were as follows. Boole was the first person to introduce an example of a non-numerical algebra and the first example of a symbolic logic [1] thus paving the way for symbolic mathematics as we know it. The familiar Boolean logic in modern notation is, however, due to Frege [2].

The Russell-Whitehead ‘Principia Mathematica’ [3] was published in 1910-3. It famously tried to develop all mathematical truths from a well-defined set of axioms and inference rules in symbolic (Boolean) logic. As authors of [4] put it, Boole wanted to study the mathematics of logic whereas Russell and Frege wanted to study logic of mathematics.

Boolean algebra is different from classical propositional calculus (algebra of sentences). This kind of distinction is too subtle to be practically useful or mathematically interesting. What is really interesting is the Lindenbaum-Tarski approach (we are going to use some the results), with a notion of equivalence of the classes of sentences. Let ψ and ϕ be two sentences in appropriate propositional language, then $\psi \equiv \phi \Leftrightarrow \psi \subset \phi$ and $\phi \subset \psi$ are both theorems.

Lindenbaum-Tarski method allows one to construct an algebra out of a classical propositional calculus and such algebra is a distributive lattice. Moreover, the approach does not only work with the classical propositional calculus. In this work we take the advantage that an algebra can be formed by a closed (according to *some* definition) set of formulas in some propositional language [5].

Wittgenstein published his ‘Tractatus Logico-Philosophicus’ [6] in 1921. While agreeing

with Russell at certain points Wittgenstein introduces a different understanding of what a mathematical object is and what is the purpose of mathematics and philosophy. In 1918 Brouwer begins the systematic intuitionistic reconstruction of mathematics with his paper *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Erster Teil, Allgemeine Mengenlehre*. (Founding Set Theory Independently of the Principle of the Excluded Middle. Part One, General Set Theory)[7].

Around the same time, Husserl developed his phenomenological approach to mathematics [8]. A detailed account of the similarities and differences of the philosophical approaches can be found in [9]. In this work, we only look at a few basic distinctions between the approaches that are relevant to the development of our formalism.

1.1.2 Phenomenological remarks

In addressing the stated goal of the work, the author follows several basic definitions. First of all, we believe that Wittgenstein's definition of the world captures the basic idea of artificial intelligence (quoted from [6]).

1.13 The facts in logical space are the world.

2.034 The structure of a fact consists of the structures states of affairs.

2.04 The totality of existing states of affairs is the world.

The next important distinction is about the nature of the states of affairs, that in our case are represented by mathematical objects.

One of the main distinctions is in the relation between the mathematical objects and time. There is more than one dichotomy that describes the relations between objects and time:

- *Static/dynamic*: an object is static exactly if at no moment are parts added to it, or removed from it. It is dynamic if at some moment there are parts added to it, or removed from it.
- *Temporal/atemporal*: an object is temporal exactly if it exists in time, and atemporal exactly if it does not exist in time.
- *Intratemporal/omnitemporal*: a temporal object is omnitemporal exactly if it is static and exists at every moment. A temporal object is intratemporal exactly if it is not omnitemporal.

A decision-making formalism that represents some model of the real world cannot be atemporal and thus the distinction between omnitemporal and intratemporal becomes important. The distinction can be represented by the three views:

1. All mathematical objects are omnitemporal. (Husserl)
2. No mathematical objects are omnitemporal. (Brouwer)
3. Some mathematical objects are omnitemporal, some are not.

The purpose of our research is to develop a formalism useful for inference within the framework of the Dempster-Shafer theory or theory of beliefs. One of the fundamental premises of the theory of beliefs is the possibility to learn and incorporate new knowledge into the already known, we also want some uniformity of our objects ruling out the third view. Brouwer's view is the most attractive for our purposes.

Brouwer's philosophical views led him to the development of his own system of mathematical foundations that he called intuitionist mathematics. Brouwer further elaborates on his view of mathematical objects: "In intuitionist mathematics a mathematical entity is not necessarily pre-determinate, and may, in its state of free growth, at some time acquire a property it did not possess before" [9].

Intuitionism is often viewed within a broader constructivist approach to mathematics. Constructivists, however, need not to accept the idea of dynamic objects. The objects that interest us are dynamic and we restrict our outlook only to intuitionist mathematics.

Brouwer separates mathematics into old formalism, pre-intuitionism (Borel, Lebesgue, Poincaré) and new formalism. The intuitionism has two acts.

First act of intuitionism completely separates mathematics from the phenomena of language described by theoretical logic, recognizing that intuitionist mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time.

Second act of intuitionism admits two ways of creating new mathematical entities: firstly in the shape of more or less freely proceeding infinite sequences of mathematical entities previously acquired (e.g. infinite decimal fractions); secondly in the shape of mathematical species, i.e. properties supposable for the mathematical entities previously acquired, satisfying the condition that if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be 'equal' to it, definitions of equality having to satisfy the conditions of symmetry, reflexivity and transitivity [10].

While commenting on the first act of intuitionism Brouwer introduces the notion of a *fleeing property* f :

- (i) for each natural number n it can be decided whether or not n possesses the property f ;

- (ii) no way of calculating a natural number n possessing f is known;
- (iii) the assumption that at least one natural number possesses f is not known to be an absurdity.

The notion of the fleeing property leads to the rejection of the tertium non datur principle and raising the problem of interpreting and intuiting the continuum, solved by the second act. At the same time the second act weakens the restrictions of the first act, while for any proposition p we have that $p \vee \neg p$ is true only if p is decidable, it follows from the second act that $\neg p \vee \neg\neg p$ is provable (absurdity or absurdity of absurdity in Brouwer's words).

In phenomenological terms Brouwer's approach is an example of a strong revisionism that has the potential of both limiting and extending actual practice.

1.2 Implications on the formalism preference

We are now ready to see how Brouwer's revisionist approach influenced the actual development of intuitionist formalism and what is relevant for the current work. In the discussion below, we do not use Brouwer's original notation or actual formal framework. Instead, we refer to Heyting's interpretation of intuitionism that is more familiar for the contemporary reader.

The familiar Boolean algebra serves as a starting point for different revisionist approaches to mathematics. In many cases, the easiest way of defining a new logic is through its relationship with Boolean algebra. We start our exposition with reminding the definition to the reader. For the sake of consistency all formal logic definitions are according to [11].

1.2.1 Boolean Logic

By Boolean logic we mean a familiar system of a set A supplied with binary operators \vee , \wedge and \rightarrow , one unary operator \neg and a constant \perp . Set A with the connectives and punctuation marks is a language \mathcal{L} . Set A is then a set of variables of \mathcal{L} , $\mathbf{Var}\mathcal{L}$, variables and constants are used to build inductively defined formulas in the set $\mathbf{For}\mathcal{L}$. In classical Boolean logic, for any elements $p_0, p_1 \in \mathbf{Var}\mathcal{L}$ we then have ten true propositions called axioms of Boolean logic.

$$(A1) \quad p_0 \rightarrow (p_1 \rightarrow p_0);$$

$$(A2) \quad (p_0 \rightarrow (p_1 \rightarrow p_2)) \rightarrow ((p_0 \rightarrow p_1) \rightarrow (p_0 \rightarrow p_2)),$$

$$(A3) \quad p_0 \wedge p_1 \rightarrow p_0;$$

$$(A4) \ p_0 \wedge p_1 \rightarrow p_1;$$

$$(A5) \ p_0 \rightarrow (p_1 \rightarrow p_0 \wedge p_1);$$

$$(A6) \ p_0 \rightarrow p_0 \vee p_1;$$

$$(A7) \ p_1 \rightarrow p_0 \vee p_1;$$

$$(A8) \ (p_0 \rightarrow p_2) \rightarrow ((p_1 \rightarrow p_2) \rightarrow (p_0 \vee p_1 \rightarrow p_2));$$

$$(A9) \ \perp \rightarrow p_0;$$

$$(A10) \ p_0 \vee (p_0 \rightarrow \perp).$$

The inference rules are:

Modus ponens: given formulas ϕ and $\phi \rightarrow \psi$ obtain ψ .

Substitution: given a formula ϕ obtain ϕs ,

where s , a substitution, is a map from $\mathbf{Var}\mathcal{L}$ to $\mathbf{For}\mathcal{L}$ defined inductively: $ps = s(p)$ for every $p \in \mathbf{Var}\mathcal{L}$, $\perp s = \perp$ and $(\psi \odot \phi)s = \psi s \odot \phi s$, for $\odot \in \{\vee, \wedge, \rightarrow\}$.

Boolean logic as defined above is denoted **Cl**. It is immediately clear that **Cl** cannot be used as a reasoning framework within the realm of intuitionist mathematics (axiom 10 is not necessary true in intuitionist mathematics). Boolean logic is the logic of atemporal objects.

1.2.2 Intuitionistic Logic

Intuitionist view of mathematics rejects the notion of a non-constructive proof and thus the axiom 10 of classical Boolean logic; so, a different logic formalism is then needed.

Moreover, we already established that Boolean logic is a logic of atemporal objects. Intuitionist mathematics takes the epistemic aspect of the truth into account: the truth of a proposition may not be known a priori, but can be learned later. Considering the possibility of *learning* new things requires a richer semantics than the one of the Boolean logic. Learning new things is reflected through the concept of a possible world or a different state of the world.

World may have different states at which different things are known. Hence, the same variable can be instantiated at some worlds not at some others. The worlds are related through the accessibility relation meaning that the knowledge in the related worlds is non-contradictory. Non-contradictory knowledge in this case means that if a variable was instantiated to some value, this

value cannot be changed at a later stage: in a world accessible from the one where the variable was first instantiated.

The truth of propositions is then established according to intuitionist understanding:

- $\phi \wedge \psi$ is true at a state (world) x if both ϕ and ψ are true at x .
- $\phi \vee \psi$ is true at a state (world) x if either ϕ or ψ is true at x .
- $\phi \rightarrow \psi$ is true at a state (world) x if for every subsequent possible state y , ϕ is true at y if and only if ψ is true at y .
- \perp is true nowhere [11].

Boolean logic **CI** then becomes an intuitionistic logic that consists of all formulas true at a single state (world) x . By intuitionistic propositional logic **Int** in a language \mathcal{L} we then understand the set of formulas that are true in all worlds and all possible configurations of such worlds.

Int is a weaker logic than **CI**: it is known that $\mathbf{Int} \subseteq \mathbf{CI}$. It is possible to embed **CI** into **Int** though. The properties of **Int** are quite well-known and we are not going to spend any time discussing them in this work.

1.2.3 Modal Logic

Another alternative to the atemporal view of **CI** is given by means of the modal logic. By modal logic we mean any logic whose semantics is not restricted to a single node and that has an operator that is capable of expressing modality (through the means of unary operators).

A modal language \mathcal{ML} can be produced from language \mathcal{L} by adding a new unary operator \Box and a new formula formation rule

- if $\phi \in \mathbf{For}\mathcal{ML}$ then $\Box\phi$ is also a formula in \mathcal{ML} .

Traditionally two unary operators are used: the *necessity* \Box and the *possibility* \Diamond . A modal logic is called a classical modal logic if possibility and necessity are related through

$$\begin{aligned} \Diamond p &\leftrightarrow \neg\Box\neg p; \\ \Box p &\leftrightarrow \neg\Diamond\neg p. \end{aligned} \tag{1.1}$$

The modality gives rise to an extra inference rule: from p conclude $\Box p$.

Different definitions of the modality operators are possible, but the one above is the most widely used. When we refer to a modal logic we mean exactly a logic that has \diamond and \square unary operators related through equations 1.1.

There is a variety of different modal logics: temporal logic, deontic logic, epistemic logic and so on which owe their existence to the meaning of the modality that is expressed through the corresponding operator. Defining, interpreting and formalizing modality is an amazing field in itself, but we are not going to venture into it at all. Instead, we look at the semantic implications of having a modal operator.

Just as in the case with intuitionistic logic we admit the possibility of gaining new knowledge at different states of the world. As in the intuitionistic case gaining new knowledge equates instantiating new variables. In this case the minimal possible modal logic will be the logic that contains all the formulas of \mathcal{ML} that are true at all worlds in all possible configurations of the universe. Such logic is called **K** and it is known that it is still a stronger logic than **CI**: $\mathbf{CI} \subseteq \mathbf{K}$, the proof of this fact can be found in many places, we quote it according to [11].

A modal logic is therefore unsuitable for building an intuitionist framework for the Dempster-Shafer theory, at least in the case when one chooses to follow Brouwer's approach.

1.2.4 Superintuitionistic Logics

Above we took a brief look at three important logic formalisms: Boolean Logic **CI**, Intuitionistic logic **Int** and Modal Logic **K**. Among the three only the intuitionistic one does not contradict the basic premises of intuitionist mathematics that we chose to follow. We have seen also that these three logics can be ordered as $\mathbf{Int} \subseteq \mathbf{CI} \subseteq \mathbf{K}$. Such ordering reflects only the first half of the definition of a strong revisionist approach (see page 5). The second half that mentions the potential of extension of the existing practice is realized through the superintuitionistic logic.

Definition 1.2.1 *A superintuitionistic or si-logic in the language \mathcal{L} is any set $L \subseteq \mathbf{For} \mathcal{L}$ that satisfies the following conditions*

1. $\mathbf{Int} \subseteq L$;
2. *L is closed under modus ponens: $\phi \in L$ and $\phi \rightarrow \psi \in L$ imply $\psi \in L$ for every $\phi, \psi \in \mathbf{For} \mathcal{L}$;*
3. *L is closed under uniform substitution: $\phi \in L$ implies $\phi s \in L$, for every $\phi \in \mathbf{For} \mathcal{L}$ and every substitution s .*

According to the definition above **For** \mathcal{L} is a si-logic; it is also clear that such logic is of little use and thus the name: the inconsistent si-logic. For any consistent si-logic L we then have $\mathbf{Int} \subseteq L \subseteq \mathbf{Cl}$.

We referred to all configurations of possible worlds while defining both **K** and **Int**, by a ‘configuration’ in this case we mean the accessibility relationship among such worlds. The notion of accessibility among the possible worlds as well as the validity/invalidity of a formula in different worlds is best captured through the idea of a semantic model or Kripke model that will be formally discussed in the due course. Up to this moment we only mentioned that both **K** and **Int** are the sets of formulas that are valid in *all* respective models.

In this work, we demonstrate that the Dempster-Shafer theory gives rise to a *specific* class of semantic models, and thus the logic that is induced by it is larger than **Int**. We thus represent the Dempster-Shafer theory through the means of a si-logic, L_{DS} . The rest of the work is devoted to building such logic and analyzing its properties.

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